Review

2017 Final Q5: $\to$ Stock $\leftarrow$ $GBM(\alpha, \sigma)$

$\to$ M.M. $\leftarrow$ interest rate r

$$V(T) = \min\{(S(T)-K_1)^+, K_2\}.$$

$\uparrow$ payoff of a sec. Find price.

$V(t) =$ AFP at time t.

RNP : $V(t) = \tilde{E}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}_t\right)$.

Under $\tilde{P}$, $S = GBM(r, \sigma)$

$$S(T) = S(t)\cdot \exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}(T)-\tilde{W}(t)\right)\right).$$

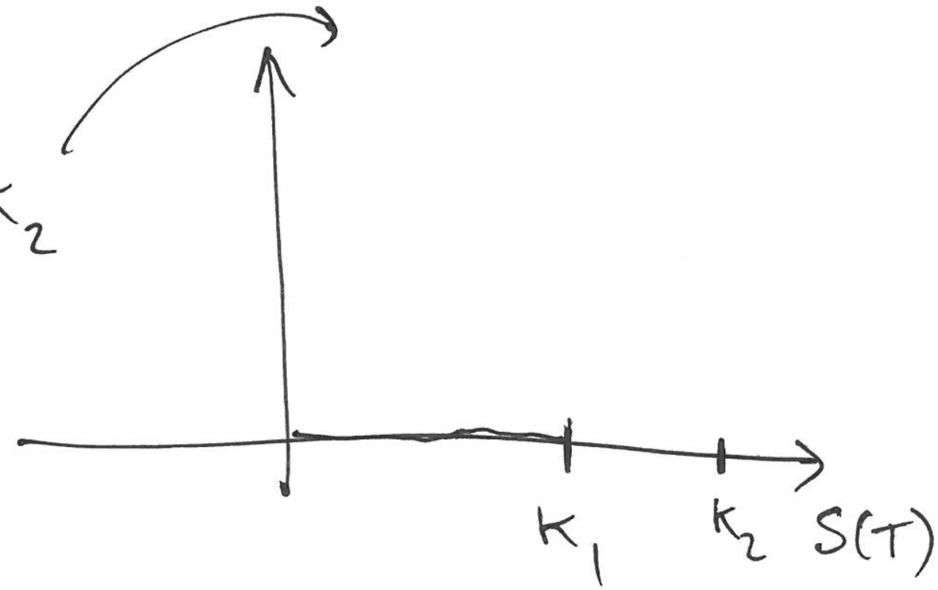
$$\Rightarrow \tilde{E}\left(e^{-r\tau}\left(S(t)\,e^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\frac{\left(\tilde{W}(T)-\tilde{W}(t)\right)\cdot\sqrt{\tau}}{\sqrt{\tau}}} - K_1\right)^+ \wedge K_2 \,\middle|\, \mathcal{F}_t\right)$$

$$\overset{\text{ind Lemma}}{=} \int \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, e^{-r\tau}\left(S(t)\,e^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y} - K_1\right)^+ \wedge K_2 \, dy.$$

$\uparrow$

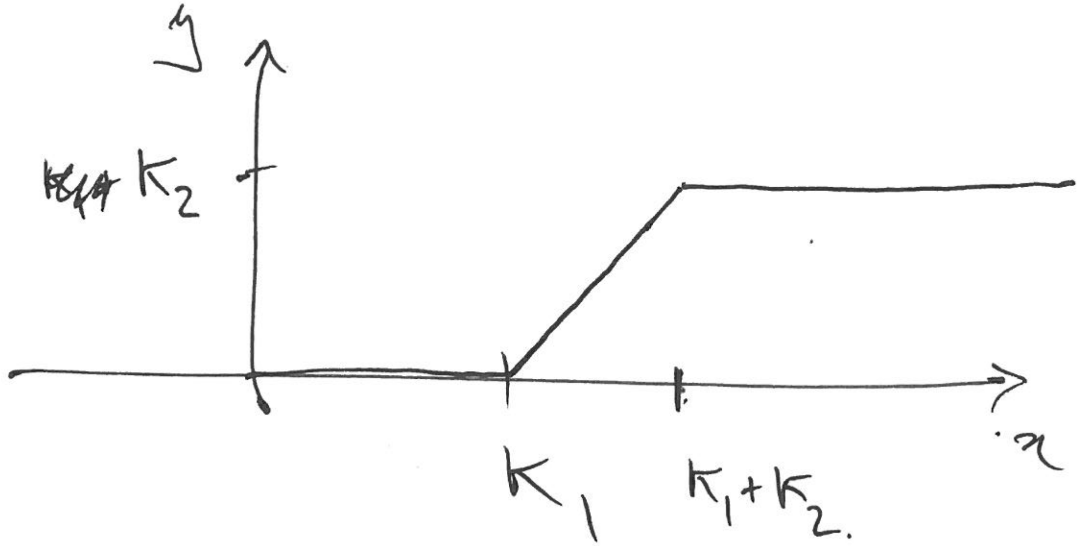
Stupid way

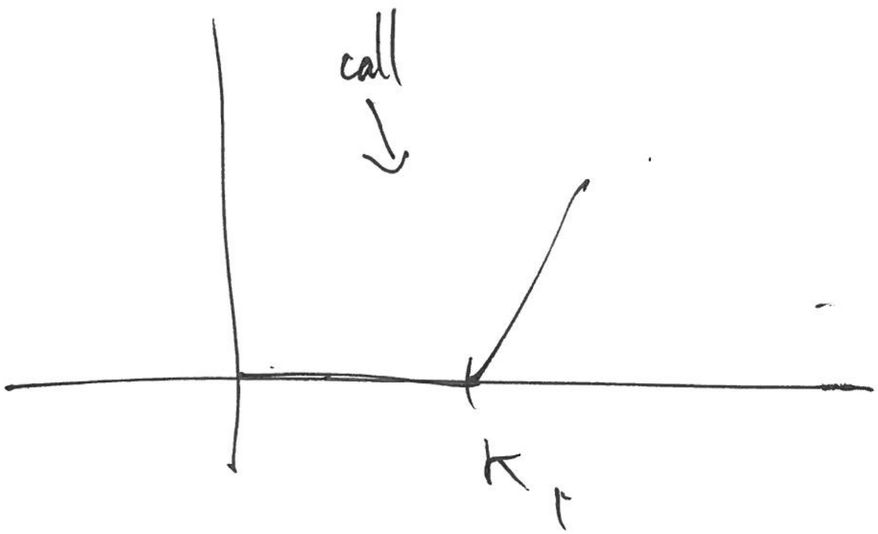
Faster way: $(S(T)-K_1)^+ \wedge K_2$



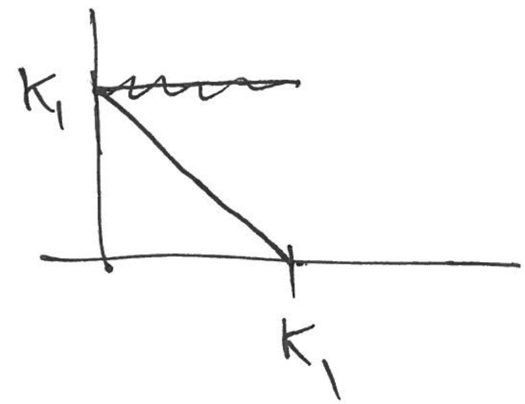
$S(T) = x$

$y = (x - K_1)^+ \wedge K_2$.

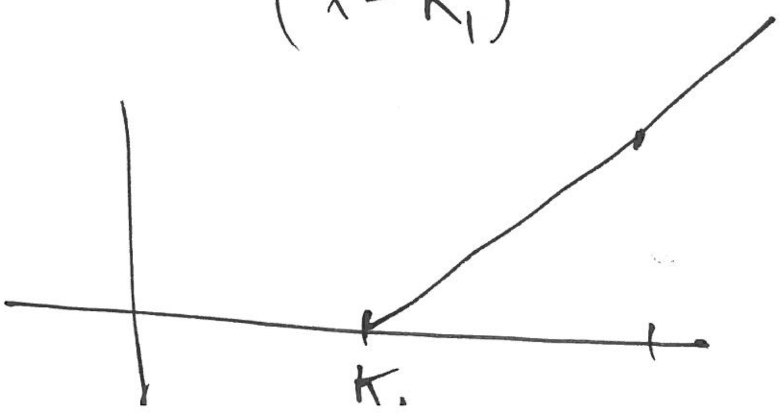




$(x - K_1)^+$

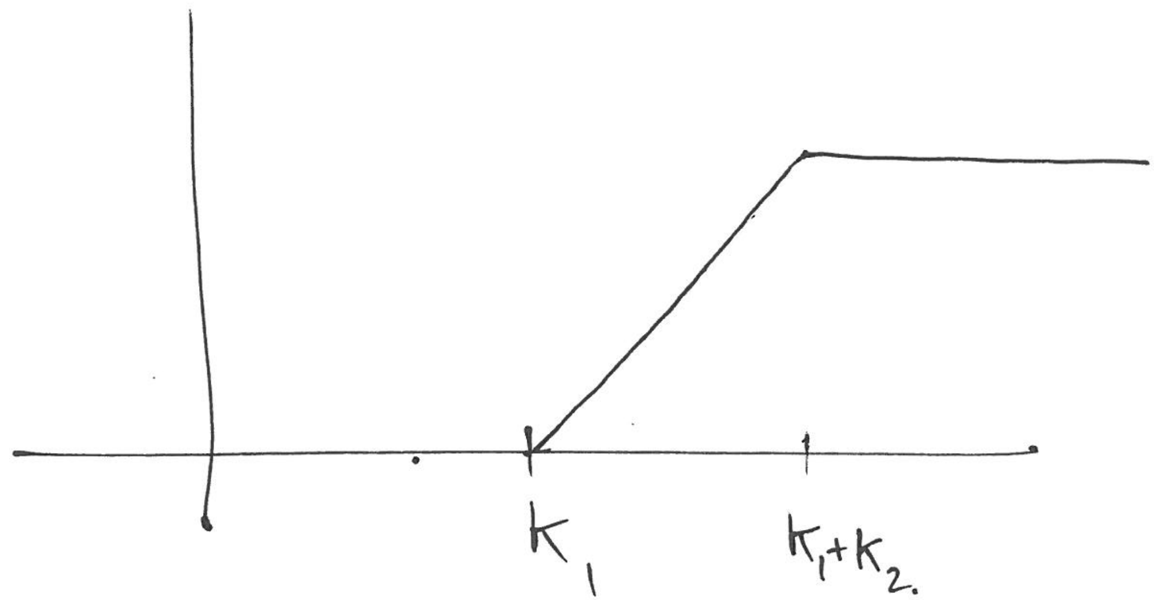


$(K_1 - x)^+$



long 1 call strike K_1

short 1 call strike $K_1 + K_2$



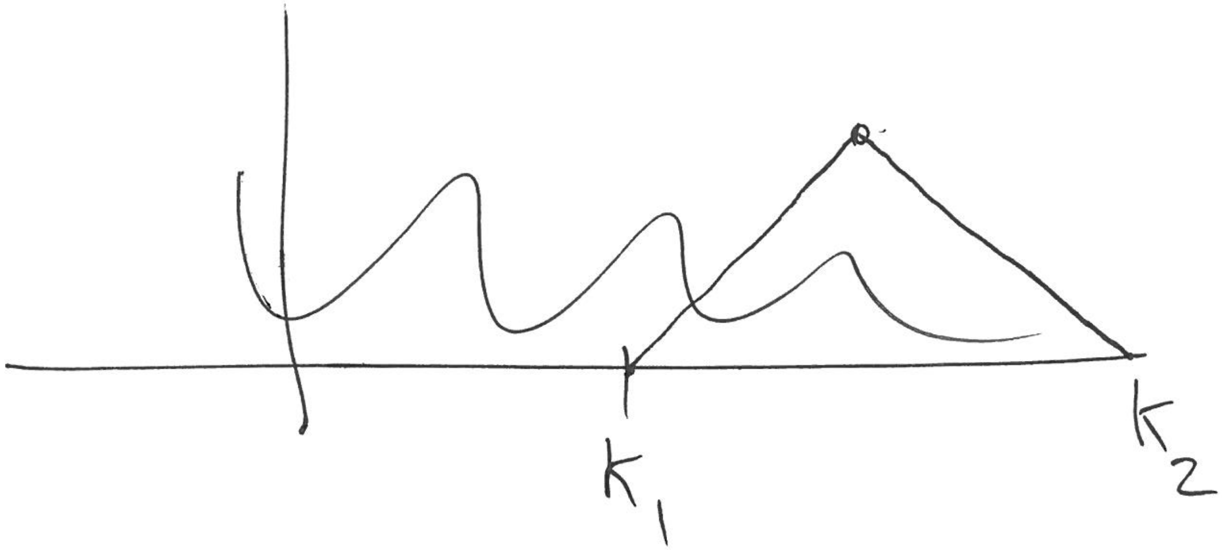
$V(T)$ = Value of a P/f long 1 call strike K_1

short 1 call " $K_1 + K_2$.

$$= c(T, S(T), K_1) - c(T, S(T), K_1 + K_2).$$

$$\Rightarrow V(t) = c(t, S(t), K_1) - c(t, S(t), K_1 + K_2).$$

B.S. formula



2019 – Final Q 5: Find a cts mg M

$M(0) = 1$ & the process $(W(t) + t^2) \cdot M(t)$ is a mg.

Option 1: Product Rule.

Option 2: Girsanov!!

Option 1: $X(t) = (W(t) + t^2)\, M(t)$

Want X to be a mg.

$$dX = (W(t)+t^2)\, dM + M\,(dW + 2t\, dt) + d[M, W]$$

Need $d[M, W] = -2M \cdot t \cdot dt$

Say $M(t) = 1 + \int_0^t \sigma(s)\, dW(s)$

$$dX = (W(t) + t^2)\, \sigma\, dW + M\, dW + 2M\, t \cdot dt + \sigma(t)\, dt$$

$dM = \sigma\, dW$

$dN = \tau\, dW$

$\Rightarrow d[M,N] = \sigma\tau\, dt$

$$d[M,W] = \sigma\, dt \overset{\text{Want}}{=} -2M \cdot t\, dt$$

$$\Rightarrow \sigma(t) = -2M(t) \cdot t$$

$$dM = \sigma(t)\, dW = -2t M\, dW$$

$$\underbrace{\frac{dM}{M}}_{\text{hopefully ``}d \ln M\text{''}} = -2t\, dW$$

Can't directly integrate.

Compute $d(\ln M) = \frac{1}{M} dM + \frac{1}{2}\left(\frac{-1}{M^2} d[M,M]\right).$

$$= -2t\, dW - \frac{1}{2M^2} \sigma^2 dt$$

$$= -2t\, dW - 2t^2\, dt$$

$$\Rightarrow \ln M(t) - \ln M(0) = -2\int_0^t s\,dW(s) - 2\int_0^t s^2\,ds.$$

$$\boxed{\Rightarrow M(t) = \underbrace{M(0)}_{1} \exp\left(-2\int_0^t s\,dW(s) \cancel{-\ \frac{2t^3}{}} - \frac{2t^3}{3}\right).}$$

Option 2:. Make $(W(t)+t^2)\cdot Z(t)$ a mg.

Girsanov: $\tilde{W}(t) = W(t) + t^2$ $\quad \boxed{d\tilde{W} = 2t\,dt + dW(t)}$

$$d\tilde{P} = Z(T)\,dP. \quad Z(T) = \exp\left(-\int_0^t 2s\,dW(s) - \frac{1}{2}\int_0^t 4s^2\,ds\right)$$

X is a mg under $\tilde{P}$

$\iff \underline{ZX}$ is a mg under P

Know $\tilde{W}$ is a mg under $\tilde{P}$.

$\Rightarrow Z\tilde{W}$ is a mg under P

$\Rightarrow Z(t)\left(W(t)+t^2\right)$ is a mg under P.

Choose $M(t) = Z(t)$ & done!

Q7 : Intrest rate $R(t) = r_0 + \theta t + \kappa \tilde{B}(t)$.

$r_0, \kappa > 0$, $\theta \in \mathbb{R}$, $\tilde{B} \to$ BM under $\tilde{P}$ (RNM).

Bond : Pays. \$1 at time T.

Q : Find AFP.

Know RNP : $V(t) =$ price at time $t = \tilde{E}\left(\frac{D(T)}{D(t)} 1 \,\middle|\, \mathcal{F}_t \right)$

$$D(t) = \exp\left(-\int_0^t R(s)\, ds\right).$$

$$V(t) = \tilde{E}\left(\exp\left(-\int_t^T R(s)\, ds\right) 1 \,\middle|\, \mathcal{F}_t \right).$$

$$= \tilde{E}\left(\exp\left(- r_0 (T-t) - \frac{\theta}{2}(T^2 - t^2) - \kappa \int_t^T \tilde{B}(s)\, ds \right) \Big| \mathcal{F}_t \right).$$

$$\cancel{= e^{-r_0(T-t) - \frac{\theta}{2}(T^2-t^2)}\, \tilde{E} \exp\left(-\kappa \int_0^T \tilde{B}(s)\right.}$$

At time 0:

$$V(0) = r\, e^{-r_0 T - \frac{\theta}{2} T^2}\, E \exp\left(-\kappa \int_0^T \tilde{B}(s)\, ds\right)$$

$\int_0^T \tilde{B}(s)\, ds$ is Normal $\longrightarrow \mathrm{Var} = \int_0^T \int_0^T (r \wedge s)\, dr\, ds. = \sigma^2$

$$V(0) = e^{-r_0 T - \frac{\theta T^2}{2}}\, E \exp\left(-\kappa \sigma N(0,1)\right) = e^{-r_0 T - \frac{\theta T^2}{2}}\, e^{+\frac{\kappa^2 \sigma^2}{2}}$$

Q2 2019 Final: $W, B \to 2$ ind std BM's.

$$X(t) = W(t)^2 + W(t)B(t) + t^2$$

Find $[X,X]$ & $[X,W]$.

Formula: $\left.\begin{aligned} dX &= \sigma\, dM \\ dY &= \tau\, dN \end{aligned}\right\} \Rightarrow d[X,Y] = \sigma\tau\, d[M,N].$

$$dX = 2W\,dW + \frac{1}{2}\cdot 2\,dt + 2t\,dt + W\,dB + B\,dW + \overbrace{d[W,B]}^{0}.$$

$$dW = 1\,dW$$

$$d[X,W] = 2W\,dt + 0 + 0 + 0 + B\,\underbrace{d[W,W]}_{dt} + 0$$
$$= (2W + B)\,dt$$

Review Q 4.5.

$x_0, \mu, \theta, \sigma \in \mathbb{R}$. $dX = \theta(\mu - X)\,dt + \sigma\,dW$.

(a) Find X

(b) $EX(t)$ & $\mathrm{cov}(X(t), X(s))$.

(a) Compute $d\left(e^{\theta t} X(t)\right) = e^{\theta t} dX + \theta e^{\theta t} X\,dt + 0$

$$= e^{\theta t}\theta(\mu - X)\,dt + e^{\theta t}\sigma\,dW + \theta e^{\theta t} X\,dt$$

$$= \theta e^{\theta t}\mu\,dt + e^{\theta t}\sigma\,dW$$

$\Rightarrow$ $e^{\theta t} X(t) = x_0 + \mu \int_0^t \theta e^{\theta s}\,ds + \sigma \int_0^t e^{\theta s}\,dW(s)$.

$$X(t) = e^{-\theta t} x_0 + \mu\theta \int_0^t e^{-\theta(t-s)}\, ds + \sigma e^{-\theta t} \cdot \int_0^t e^{\theta s}\, dW(s)$$

4.7 (Review). $\theta \in \mathbb{R}$

$$Z(t) = \exp\left(\theta W(t) - \frac{\theta^2 t}{2}\right).$$

Given $s < t$, some fn f.

Want $\underline{E}\left(f(Z(t)) \,\middle|\, \mathcal{F}_s \right) \overset{\text{want}}{=\!=} g(Z(s))$ for some fn g.

(Find g).

$$E\left(f(Z(t))\,\middle|\,\mathcal{F}_s\right) = E\left(f\left(e^{\theta W(t) - \frac{\theta^2}{2}t}\right)\,\middle|\,\mathcal{F}_s\right).$$

$$= E\left[f\left(e^{\theta(W(t)-W(s)) - \theta^2/2\, t + \theta W(s)}\right)\,\middle|\,\mathcal{F}_s.\right].$$

$$\underset{\text{indep Lema}}{=} \int_{y\in\mathbb{R}} \frac{e^{-y^2/t-s}}{\sqrt{2\pi(t-s)}} f\left(e^{\theta y - \theta^2/2\, t + \theta W(s)}\right). \, dy$$

$$= \int \frac{e^{-y^2/(t-s)}}{\sqrt{2\pi(t-s)}} f\Big(\underbrace{e^{\theta W(s) - \frac{\theta^2 s}{2}}}_{Z(s)} \cdot e^{\theta y - \frac{\theta^2}{2}(t-s)}\Big)\, dy$$

Let $g(x) = \int\limits_{y \in \mathbb{R}} \frac{e^{-y^2/(t-s)}}{\sqrt{2\pi(t-s)}} f\left(x\, e^{\theta y - \frac{\theta^2}{2}(t-s)}\right) dy$.

2019 Q3 Final

Find $E(W(s) \mid W(t))$ $(s < t)$.

$= E\left(W(s) \mid W(t) - W(s) + W(s) \mid \right)$

$= \int\limits_{x \in \mathbb{R}} \int\limits_{y \in \mathbb{R}} x \mid y + x \mid \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \frac{e^{-y^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \, dx\, dy$

① $\int_0^t f(s)\,W(s)\,ds$ is Normal as long as f is NOT RANDOM.

② $\int_0^t g(s)\,dW(s)$ " " g " " ".