Today:

- Risk neutral pricing
  - ↳ Girsanov's Theorem
  - ↳ Lévy's theorem

NO Office Hour on Wednesday.

Instead I will have OH on Tuesday

10 AM – 11 AM NY Only on Canvas by Request

11 AM – 12 PM Pittsburgh Students in WEH 7211.

Claim 1: Let W_t^1 and W_t^2 be BM's with correlation $\rho \in [-1,1]$.

$$d[W^1, W^2]_t = \rho \, dt$$

then $B_t := \dfrac{W_t^1 - \rho W_t^2}{\sqrt{1-\rho^2}}$ is a BM

that is independent of W^2.

pf: Lévy's Theorem:

$B_0 = 0$ ✓ B_t continuous martingale ✓
↳ as a sum of two martingales.

$$[B,B]_t = [W^1 - \rho W^2, W^1 - \rho W^2]_t \cdot \left(\frac{1}{1-\rho^2}\right)$$

$$= \left(\frac{1}{1-\rho^2}\right)\left([W^1,W^1]_t - 2\rho [W_t^1, W^2]_t + \rho^2 [W^2,W^2]_t\right)$$

$$= \frac{1}{(1-\rho^2)}\left(t - 2\rho^2 t + \rho^2 t\right)$$

$$= \frac{1}{(1-\rho^2)} \times t \times [1 - \rho^2] = t \quad \checkmark$$

Levy $\Rightarrow$ B is a BM.

B indepent of W^2 $\Leftrightarrow$ $[B, W^2]_t = 0$

Because they are Brownian Motions.

$$[B, W^2]_t = \frac{1}{\sqrt{1-\rho^2}} \left([W^1, W^2]_t - \rho [W^2, W^2]_t \right)$$

$$= \frac{1}{\sqrt{1-\rho^2}} (\rho t - \rho t) = 0.$$

View (B, W^2) as a 2-dimensional BM.

If $[B, W^2] = 0$ then we have

$$[B, B]_t = t$$

$$[B, W^2]_t = 0$$

$$[W^2, W^2]_t = t$$

So Lévy $\Rightarrow$ (B, W^2) is a standard 2-dimensional BM and so in particular are independent. $\square$

Remark: If I start with two Brownian motions B and W^2 which are independent

and I define

$$W_t^1 := \sqrt{1-\rho^2}\, B_t + \rho W_t^2$$

then W_t^1 has correlation ρ with W_t^2.

Notice $[W^1, W^2]_t = \rho t$

$[W^1, W^1]_t = [W^2, W^2]_t = t.$

$$\frac{[W^1, W^2]_t}{\sqrt{[W^1, W^1]_t} \cdot \sqrt{[W^2, W^2]_t}} = \frac{\rho t}{\sqrt{t} \cdot \sqrt{t}} = \rho$$

Recall: Girsanov's Thm

$b_t = (b_t^1, \dots, b_t^d)$.

$W_t = (W_t^1, \dots, W_t^d) \rightsquigarrow$ BM under $\mathbb{P}$. (standard)

Define $Z_t = \exp\left(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t \|b_s\|^2 ds\right)$.

If $\mathbb{E}[Z_t] = 1 \ \forall t$ we can define a new measure

$\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P}$.

(i.e. $\tilde{\mathbb{P}}(A) = \underset{\text{under } \mathbb{P}}{\mathbb{E}}[\mathbb{1}_A Z_T]$)

then $\tilde{W}_t := W_t + \int_0^t b_s \, ds$ is a BM under $\tilde{\mathbb{P}}$.

If $dX_t = \alpha_t \, dt + \sigma_t \, dW_t$ is Ito decomp under $\mathbb{P}$.

$= \alpha_t \, dt + \sigma_t (d\tilde{W}_t - b_t \, dt)$

$= (\alpha_t - \sigma_t b_t) dt + \sigma_t \, d\tilde{W}_t$

is the Ito decomp under $\tilde{\mathbb{P}}$.

Chapter 4 lemma 1.11.

If X is $\mathcal{F}_T$ measurable then

$$\tilde{\mathbb{E}}[X|\mathcal{F}_t] = \frac{1}{Z_t}\mathbb{E}[Z_T X|\mathcal{F}_t]$$

Claim 2: If B and W are independent BM's under $\mathbb{P}$ and we change measure using W

(i.e. $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$ where

$$Z_T = \exp\left(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t \|b_s\|^2 ds\right))$$

then B is a BM under $\tilde{\mathbb{P}}$ and B is independent of $\tilde{W}_t = W_t + \int_0^t b_s ds$.

Pf: Levy: $B_0 = 0$ ✓ (under $\tilde{\mathbb{P}}$)

$[B,B]_t = t$

Note QV does not depend on the measure.

what about martingale?

$$\tilde{\mathbb{E}}[B_t | \mathcal{F}_s] \overset{\text{Lem}}{=} \frac{1}{Z_s} \mathbb{E}[B_t Z_t | \mathcal{F}_s].$$

Z_t is a martingale under $\mathbb{P}$.

$$dZ_t = \theta Z_t \, dW_t$$

$$d(B_t Z_t) = Z_t \, dB_t + B_t \, dZ_t + [B,Z]_t$$

$Z_t \, dB_t$ → martingale; $B_t \, dZ_t$ → martingale; $[B,Z]_t = 0$. ⇒ is a martingale under $\mathbb{P}$.

∴ $B_t Z_t$ is a martingale under $\mathbb{P}$.

$\therefore \quad \mathbb{E}[B_t Z_t \mid \mathcal{F}_s] = B_s Z_s$

$\therefore \quad \tilde{\mathbb{E}}[B_t \mid \mathcal{F}_s] = \dfrac{Z_s B_s}{Z_s} = B_s$ ✓

So by Lévy B is a ~~marti~~ BM under $\tilde{\mathbb{P}}$.

Independence:

$[B, \tilde{W}]_t = [B, W + \int_0^{\cdot} b_s \, ds]_t = [B, W]_t = 0$

□.

Risk-Neutral pricing:

$$dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t \quad \text{(under } \mathbb{P}\text{)}$$

We have risk free rate $R_t \rightsquigarrow$ adapted process.

$V(T)$ ~~pric~~ pay off of derivative security.

$$d\tilde{\mathbb{P}} = Z_T\,d\mathbb{P} \quad \text{where}$$

$$Z_t = \exp\left(-\int_0^t \frac{\alpha_s - R_s}{\sigma_s}\, dW_s - \frac{1}{2}\int_0^t \left(\frac{\alpha_s - R_s}{\sigma_s}\right)^2 ds\right)$$

$\frac{\alpha_s - R_s}{\sigma_s}$: market price of risk or sharpe ratio.

then the APP of V is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)\,ds}\, V(T) \mid \mathcal{F}_t\right]$$

when $R(s) = r$ (constant)

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\, V(T) \mid \mathcal{F}_t\right]$$

under $\tilde{\mathbb{P}}$ $e^{-\int_0^t R(s)ds} S_t$ is a martingale.

In Black-Scholes $e^{-rt} S_t$ is a martingale!

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t \quad \text{under } \tilde{\mathbb{P}}.$$

<u>Ex</u> (Magrabe option).

Two risky assets.

$$\begin{cases} dS_t^1 = \alpha_1 S_t^1 dt + \sigma_1 S_t^1 dW_t^1 \\ dS_t^2 = \alpha_2 S_t^2 dt + \sigma_2 S_t^2 dW_t^2 \end{cases} \quad (\text{under } \mathbb{P}).$$

$$d[W^1, W^2]_t = \rho dt$$

Constant interest rate r.

The Magrabe option allows its holder to exchange b assets of S^2 for a assets of S^1 at time $T>0$ where $a, b > 0$.

Pay off is $V(T) = (aS_T^1 - bS_T^2)^+$

"real option"

S_t^1 is natural gas and S_t^2 is electricity

we want to find

$$V_t = \tilde{\mathbb{E}}\left[e^{-r(T-t)}(aS_T^1 - bS_T^2)^+ \mid \mathcal{F}_t\right]$$

$$= \tilde{\mathbb{E}}\left[e^{-r(T-t)} a S_T^2 \Big(\underbrace{\frac{S_T^1}{S_T^2}}_{Y_T} - \frac{b}{a}\Big)^+ \mid \mathcal{F}_t\right]$$

~~$d(Y_t) = \ldots$~~

$$\begin{cases} dS_t^1 = rS_t^1\,dt + \sigma_1 S_t^1\,d\tilde{W}_t^1 \\ dS_t^2 = rS_t^2\,dt + \sigma_2 S_t^2\,d\tilde{W}_t^2 \end{cases} \quad (\text{under } \tilde{\mathbb{P}})$$

$$d[\tilde{W}^1, \tilde{W}^2]_t = \rho\,dt$$

(last time)

$$d(Y_t) = Y_t(\cancel{r} - \cancel{r} - \sigma_1\sigma_2\rho + \sigma_2^2)\,dt + \sigma_1 Y_t\, d\tilde{W}_t^1 - \sigma_2 Y_t\, d\tilde{W}_t^2$$

How to compute

$$\tilde{\mathbb{E}}\left[S_t^2\left(Y_t - \tfrac{b}{a}\right)^+ \middle| \mathcal{F}_t\right] ?$$

$e^{-rt}S_t^2$ is a martingale under $\tilde{\mathbb{P}}$

$$\| \quad \cancel{e^{-rt}}\, S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)t + \sigma_2 \tilde{W}_t^2}$$

$$\| \quad S_0^2 \underbrace{e^{\int_0^t \sigma_2\, d\tilde{W}_s^2 - \frac{1}{2}\int_0^t \sigma_2^2\, ds}}_{Z_t}.$$

We will switch measure with Z_T!

define $d\hat{\mathbb{P}} = Z_T\, d\tilde{\mathbb{P}}$

By lemma 1.11

$$\hat{\mathbb{E}}\left[(Y_T - \tfrac{6}{a})^+ \mid \mathcal{F}_t\right] = \frac{1}{Z_t}\tilde{\mathbb{E}}\left[Z_T (Y_T - \tfrac{6}{a})^+ \mid \mathcal{F}_t\right].$$

$$\left(Z_t = \frac{e^{-rt}S_t^2}{S_0}\right) = \frac{e^{rt}S_0^2}{S_t^2}\tilde{\mathbb{E}}\left[\frac{e^{-rT}S_T^2}{S_0^2}(Y_T - \tfrac{6}{a})^+ \mid \mathcal{F}_t\right].$$

$$= \frac{e^{-r(T-t)}}{S_t^2}\tilde{\mathbb{E}}\left[S_T^2 (Y_T - \tfrac{6}{a})^+ \mid \mathcal{F}_t\right].$$

$$\therefore V_t = a S_t^2 \hat{\mathbb{E}}\left[(Y_T - \tfrac{6}{a})^+ \mid \mathcal{F}_t\right].$$

by subbing in.

what are the dynamics of Y_t under $\hat{\mathbb{P}}$.

$\hat{W}_t^2 := \tilde{W}_t^2 - \sigma_2 t$ is a BM under $\hat{\mathbb{P}}$.

$$dY_t = (-\sigma_1\sigma_2\rho Y_t + \sigma_2^2 Y_t)dt + \sigma_1 Y_t d\tilde{W}_t^1 - \sigma_2 Y_t d\tilde{W}_t^2$$

$$= (-\sigma_1\sigma_2 \rho Y_t)dt + \sigma_1 (Y_t d\tilde{W}_t^1) - \sigma_2 Y_t \underbrace{(d\tilde{W}_t^2 - \sigma_2 dt)}_{d\hat{W}_t^2}.$$

$$= -\sigma_1\sigma_2\rho Y_t dt + \sigma Y_t d\tilde{W}_t^1 - \sigma_2 Y_t d\hat{W}_t^2.$$

By Claim 1 we can write

$$\tilde{W}_t^1 = \sqrt{1-\rho^2} B_t + \rho \tilde{W}_t^2$$ where B is a BM independent of $\tilde{W}_t^2$ under $\tilde{\mathbb{P}}$.

(*)
$$= \sqrt{1-\rho^2} B_t + \rho(\hat{W}_t^2 + \sigma_2 t)$$
$$= \rho\sigma_2 t + \sqrt{1-\rho^2} B_t + \rho \hat{W}_t^2$$

claim 2 $\Rightarrow$ B is independent of $\hat{W}_t^2$ under $\hat{\mathbb{P}}$.

by Claim 1: $\hat{W}_t^1 := \sqrt{1-\rho^2} B_t + \rho \hat{W}_t^2$ is a BM under $\hat{\mathbb{P}}$ with correlation ρ with $\hat{W}_t^2$

$\therefore$ by (*). $\hat{W}_t^1 = \tilde{W}_t^1 - \rho\sigma_2 t$

$dY_t = -\sigma_1\sigma_2\rho Y_t\,dt + \sigma_1 Y_t\,d\tilde{W}_t^1 - \sigma_2 Y_t\,d\hat{W}_t^2$ (from before)

$= -\sigma_1\sigma_2\rho Y_t\,dt + \sigma_1 Y_t\,(d\hat{W}_t^1 + \rho\sigma_2 dt) - \sigma_2 Y_t\,d\hat{W}_t^2$

$= ~~\sigma_1 Y_t\,dW~~ \; Y_t\,(\sigma_1 d\hat{W}_t^1 - \sigma_2 d\hat{W}_t^2)$.

Let's compute $[\sigma_1 W_t^1 - \sigma_2 W^2, \sigma_1 W^1 - \sigma_2 W^2]_t$

$= \sigma_1^2 t - 2\sigma_1\sigma_2\, d[\hat{W}^1, \hat{W}^2]_t + \sigma_2^2 t$

$= (\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2)\,t$

$\sigma_1 d\hat{W}_t^1 - \sigma_2 d\hat{W}_t^2$ is a cont mart and is 0 at time 0.

above calculation shows

$$\hat{B}_t = \frac{\sigma_1 \hat{W}_t^1 - \sigma_2 \hat{W}_t^2}{\sqrt{\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2}}$$

is a BM by Lévy under $\hat{\mathbb{P}}$.

Let $\sigma := \sqrt{\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2}$

then $\boxed{dY_t = \sigma Y_t \, d\hat{B}_t}$

To summarize we have

$$V_t = a S_t^2 \hat{\mathbb{E}}\left[\left(Y_t - \frac{b}{a}\right)^+ \middle| \mathcal{F}_t\right]$$

where Y_t has dynamics under $\hat{\mathbb{P}}$ as above.

$\hat{\mathbb{E}}\left[\left(Y_t - \frac{b}{a}\right)^+ \middle| \mathcal{F}_t\right]$ = call price with $K = \frac{b}{a}$, volatility σ, $r = 0$ and $S_t = Y_t$

V_t =

$$V_t = a S_t^2 \left(Y_t N(d_1) - \frac{b}{a} N(d_2) \right) = a S_t^1 N(d_1) - b S_t^2 N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{Y_t}{b/a}\right) + \sigma^2 (T-t)}{\sigma \sqrt{T-t}} = \frac{\ln\left(\frac{a S_t^1}{b S_t^2}\right) + \sigma^2 (T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

↳ relative volality.

Exam Tips:

1) Know you main formulas

↳ Ito's formula (multivariate version).

↳ Girsanov's Theorem (multivariate version).

↳ RN pricing.
  ↳ how to compute.
  ↳ properties ⇝ e.g. Discounted stock price is martingale under $\tilde{\mathbb{P}}$.
  Market price of risk is $\frac{\alpha_t - R_t}{\sigma_t}$ . . . .

↳ Lévy's Thm.

↳ Ito Isometry

↳ Call formula for Black-Scholes & PDE.

↳ properties of Gaussians (MGF's, pdf) etc...

2) Think about if your answer makes sense.

↳ In previous years when pricing options some students solutions had e.g. a negative price OR depended on a future unknown value.

$$\mathbb{E}[X^2] = 0 \quad \text{but } X \neq 0.$$

This doesn't happen

3)