COURSE EVALS: $\xrightarrow{\text{IF}}$ $\boxed{\text{75\% response rate}}$

THEN ⇒ (1) Grades Early

(2) Review session Monday

<u>Q1:</u> $f = f(t)$ (not random).

Q: Find the dist of $X(t) = \int_0^t f(s) W(s)\, ds$.

$$EX(t) = E\int_0^t f(s) W(s)\, ds = \int_0^t E f(s) W(s)\, ds = 0$$

Option 1: Find MGF (Might work)

Find Variance: $E\,X(t)^2$ (Trick 1).

$$E\,X(t)^2 = E\left(\int_0^t f(s)\,W(s)\,ds\right)^2 \overset{\text{Ito ISOM}}{=} \cancel{E\int_0^t f(s)^2\,W(s)^2\,ds}$$

$$= E\left(\int_0^t f(s)\,W(s)\,ds\right)\left(\int_0^t f(s)\,W(s)\,ds\right)$$

$$= E\left(\int_0^t f(\tau)\,W(\tau)\,d\tau\right)\left(\int_0^t f(s)\,W(s)\,ds\right).$$

$$= E\int_{\tau=0}^{t}\int_{s=0}^{t} f(\tau)\,f(s)\,W(\tau)\,W(s)\,ds\,d\tau$$

$$= \int_0^t\int_0^t f(\tau)\,f(s)\,(\tau\wedge s)\,ds\,d\tau$$

Trick 2: Write $\int_0^t f(s)\, W(s)\, ds = g(t, W(t)) + \int_0^t h(s, W(s))\, dW(s)$.

Guess some fn $\underline{G}$ ~~d(t, w) cdt~~ $\boxed{\cancel{G' = \text{derivative}}}$

$$d\big(G(t, W(t))\big) = \overset{f(t)\,W(t)}{\cancel{f(t)}}\, dt + (\ )\, dW$$

Note $d\big(G(t, W(t))\big) = \partial_t G\, dt + \partial_x G\, dW + \frac{1}{2} \partial_x^2 G\, dt$

$$= \underbrace{\left(\partial_t G + \frac{1}{2} \partial_x^2 G\right)}_{\text{Want } = f(t)\,W(t)} dt + \partial_x G\, dW$$

Guess $G(t, x) = \cancel{tx} : F(t) \cdot x$ where $F(t) = \int_0^t f(s)\, ds$.

$$\partial_t G = x\,F'(t) = x\,f(t)$$

$$\partial_x G = F(t) \quad \& \quad \partial_x^2 G = 0$$

$$\Rightarrow d(G(t,W(t))) = f(t)W(t)\,dt + F(t)\,dW + 0$$

$$\Rightarrow G(t,W(t)) - G(0,W(0)) = \int_0^t f(s)W(s)\,ds + \int_0^t F(s)\,dW(s)$$

$$\Rightarrow f(t)W(t) = \underbrace{\int_0^t f(s)W(s)\,ds}_{} + \int_0^t F(s)\,dW(s)$$

$$\Rightarrow \int_0^t f(s)W(s)\,ds = f(t)W(t) - \int_0^t F(s)\,dW(s)$$

compute $E\left(\int_0^t f(s)W(s)\,ds\right)^2$ using ↑

$$X(t) = \int_0^t f(s)\, W(s)\, ds.$$ Know $EX(t) = 0$, $EX(t)^2 =$ ______

Dist of X?

$$\left.\underbrace{\lim_{\|P\| \to 0} \sum f(t_i)\, W(t_i)\, (t_{i+1} - t_i)}_{\text{sum of normals.}}\right\} \Rightarrow \text{normal.}$$

$$\boxed{\text{Expect dist of } X \text{ is Normal}}$$

4.2 M, N 2 mg's. $d[M,M] = \sigma(t)\,dt$ $\quad d[M,N] = \rho(t)\,dt$

$d[N,N] = \tau(t)\,dt$

$\boxed{\sigma, \tau, \rho \text{ NOT RANDOM}}$

1. find MGF
2. If $\sigma = \tau = 1$ & $\rho = 0$ then (M,N) is a std 2D B.M.

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1. MGF: $E\exp\left(\lambda M(t) + \mu N(t)\right) = \varphi(t)$ $\quad f(t,x,y) = \exp(\lambda x + \mu y)$

$$d\left(\exp\left(\lambda M(t) + \mu N(t)\right)\right) = \lambda e^{\lambda M + \mu N}\,dM + \mu e^{\lambda M + \mu N}\,dN$$
$$+ \frac{1}{2}\left(\lambda^2 \sigma + \mu^2 \tau + 2\lambda\mu\rho\right)e^{\lambda M + \mu N}\,dt$$

$$e^{\lambda M(t) + \mu N(t)} - 1 = \int_0^t (\ ) \, dM + \int_0^t (\ ) \, dN$$

$$+ \frac{1}{2} \int_0^t (\lambda^2 \sigma + \mu^2 \tau + 2\lambda\mu\rho) \, e^{\lambda M + \mu N} \, ds$$

Take Exp values:

$$\varphi(t) - 1 = 0 + 0 \quad (\because M, N \text{ are mg's}).$$

$$+ \frac{1}{2} \int_0^t (\lambda^2 \sigma + \mu^2 \tau + 2\lambda\mu\rho) \, \varphi(s) \, ds$$

$$\varphi'(t) = \frac{1}{2} \left(\lambda^2 \sigma(t) + \mu^2 \tau(t) + 2\lambda\mu\rho(t) \right) \varphi(t)$$

$$\Rightarrow \varphi(t) = \underbrace{\varphi(0)}_{1.} \cdot \exp\left(\frac{1}{2} \int_0^t (\lambda^2 \sigma(s) + \mu^2 \tau(s) + 2\lambda\mu\rho(s)) \, ds \right).$$

(2) $\sigma = 1 = \tau$ & $\rho = 0$.

$$\Rightarrow \varphi(t) = E e^{\lambda M(t) + \mu N(t)} = \exp\left(\frac{t}{2}(\lambda^2 + \mu^2)\right)$$

$$= \text{MGF of } \underbrace{N\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t\right)}_{\text{MGF } \cancel{\text{dist}} \text{ of 2D BM at time } t}.$$

$X \sim N(0,t)$, $E e^{\lambda X} = e^{\frac{\lambda^2}{2} t}$

$X \sim N(\mu, \sigma^2)$, $E e^{tX} =$

Q3

Market $\begin{cases} \to \text{M.M.} & \text{interest rate } r \\ \to \text{Stock} & \text{GBM}(\alpha, \sigma) \end{cases}$

α: mean return rate

σ: volatility.

Security: Pays $V(T) = \frac{1}{T}\int_0^T S(t)\,dt$.

Find the AFP of this security at time t.

RNP: $V(t) = \tilde{E}\left(e^{-r(T-t)} \cdot V(T) \mid \mathcal{F}_t\right)$.

$\boxed{d\tilde{P} = Z(T)\,dP \ \& \ Z(T) = \underline{\qquad\qquad}}$ ← Ignore.

Know Under $\tilde{P}$, S is ~~GMB~~ GBM(r, σ)

$\tilde{W} \to$ BM under $\tilde{P}$.

$$S(t) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}(t)\right).$$

$$V(s) = \tilde{E}\left(\frac{e^{-r(T-s)}}{T} \cdot \int_0^T S(t)\,dt \,\Big|\, \mathcal{F}_s\right)$$

$$= \frac{e^{-r(T-s)}}{T}\tilde{E}\left(\int_0^s S(t)\,dt \,\Big|\, \mathcal{F}_s\right) + \underbrace{\frac{e^{-r(T-s)}}{T}\tilde{E}\left(\int_s^T S(t)\,dt \,\Big|\, \mathcal{F}_s\right)}$$

$$= \frac{e^{-r(T-s)}}{T}\int_0^s S(t)\,dt +$$

Compute $\cancel{e^{-r(T-s)}}$ $\tilde{E}\left(\int_s^T S(t)\,dt \,\Big|\, \mathcal{F}_s\right) = \int_s^T \tilde{E}\left(S(t) \,|\, \mathcal{F}_s\right) dt$

Note $S(t) = S(s) \exp\left(\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma\left(\tilde{W}(t) - \tilde{W}(s)\right)\right)$.

$$\Rightarrow \tilde{E}\left(S(t) \,|\, \mathcal{F}_s\right) = S(s)\, e^{\left(r - \frac{\sigma^2}{2}\right)(t-s)}\, E\left(e^{\sigma\left(\tilde{W}(t) - \tilde{W}(s)\right)} \,\Big|\, \mathcal{F}_s\right).$$

$$= S(s)\, e^{\left(r - \frac{\sigma^2}{2}\right)(t-s)}\, e^{\frac{\sigma^2}{2}(t-s)} = S(s)\, e^{r(t-s)}$$

$$\Rightarrow V(s) = \frac{e^{-r(T-s)}}{T} \int_0^s S(t)\,dt + \frac{e^{-r(T-s)}}{T} \int_s^T S(s)\, e^{r(t-s)}\,dt$$

& simplify.

faster trick: Know $e^{-rt} S(t)$ is a mg under $\tilde{P}$.

$$\Rightarrow \tilde{E}\left(S(t) \mid \mathcal{F}_s \right) = e^{rt} \tilde{E}\left(e^{-rt} S(t) \mid \mathcal{F}_s \right)$$

$$= e^{rt} e^{-rs} S(s) = e^{r(t-s)} S(s).$$

4.8 $W \to$ B.M. Let $B(t) = \int_0^t \operatorname{sign}(W(s))\, dW(s)$.

(a) Is B a BM?

Yes: Use Levy: $B(0) = 0$

Need (1) B is cts (2) $d[B,B] = dt$ & (3) Mg!!

Check: ⓐ B is cts. mg (since B is an Itô integral).

ⓑ Compute $d[B,B] = \text{sign}(W(t))^2\, dt \underset{a.s.}{=} dt$

By Levy, B is a B.M..

② Is there an adapted process σ such that

$$W(t) = \int_0^t \sigma(s)\, dB(s) ?$$

Sol: Guess $\Leftrightarrow dW = \sigma(s)\, dB(s) \Leftrightarrow dB = \frac{1}{\sigma} dW$

Know $dB = \text{sign}(W(s))\, dW(s)$

Choose $\sigma(t) = \dfrac{1}{\text{sign}(W(t))} = \text{sign}(W(t))$

$$\Rightarrow W(t) = \int_0^t \sigma(s)\, dB(s) = \int_0^t \operatorname{sign}(W(s))\, \cancel{dW(s)}\, dB(s).$$

Q: Are W & B indep? Are W & B uncorrelated.

Compute $E\, W(t) B(t)$.

Itô: $$d\big(W(t)B(t)\big) = W(t)\, dB(t) + B(t)\, dW(t) + d[W,B](t).$$

$$\left.\begin{aligned} dW &= dW \\ dB &= \operatorname{sign}(W(t))\, dW(t) \end{aligned}\right\} d[W,B] = \operatorname{sign}(W(t))\, dt$$

$$\Rightarrow W(t)B(t) = \int_0^t W(s)\, dB(s) + \int_0^t B(s)\, dW(s) + \int_0^t \operatorname{sign}(W(s))\, ds.$$

$$E\, W(t)\, B(t) = 0 + 0 + \underbrace{\int_0^t E\, \text{sign}(W(s))\, ds}_{0}.$$

$\Rightarrow E\, W(t)\, B(t) = 0$. $\Rightarrow$ W & B are uncorrellated.

(W & B are not jointly normal, so this $\not\Rightarrow$ W & B are indep).

<u>Claim:</u> W & B are NOT indep.

Note If W & B are indep $\Rightarrow [W, B] = 0$

But $[W, B] = \int_0^t \text{Sign}\, W(s)\, ds \neq 0$. $\Rightarrow$ W & B NOT indep.