Girsanow Thm:

Let $b(t) = (b_1(t), \dots b_d(t))$ be a d-dim adapted. process,

$W(t)$ is a d-dim B.M under $\mathbb{P}$, let

$$\tilde{W}(t) = W(t) + \int_0^t b(s)\,ds \quad \left(d\tilde{W}(t) = b(t)dt + dW(t) \right).$$

Let $Z(t) := \exp\left(-\int_0^t b(s) \cdot dW(s) - \frac{1}{2}\int_0^t |b(s)|^2 ds\right)$

$$b(s) \cdot dW(s) = \sum_{i=1}^{d} b_i(t)\,dW_i(t) \qquad |b(s)|^2 = \sum_{i=1}^{d} b_i(t)^2$$

If $Z(t)$ is a mtg, then $\tilde{W}(t)$ is a B.M. under $\tilde{\mathbb{P}}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z(T).$$

Consider we are trading a risky asset whose price follows G.B.M (2)
with mean return α and vol σ. We have a money mkt acct
~~they~~ which has interest rate r_1 for $0 \le t \le T_1$ and r_2 for $t > T_1$.
Suppose ~~the~~ an European call expires at $T > T_1$, compute its
price at $t < T_1$ using RN pricing formula.

RN pricing: $D(t) = \exp(-\int_0^t r(s)\,ds)$. payoff: $H(X(T))$

$$V(t) = \tilde{\mathbb{E}}[D(t) H(X(T)) \mid \mathcal{F}_t].$$

$\tilde{\mathbb{E}}$ is under RN measure $\tilde{\mathbb{P}}$.

Let $r(t) = r_1 \mathbb{1}_{\{t \le T_1\}} + r_2 \mathbb{1}_{\{t > T_1\}}$ $\quad D(t) = \exp\left(-\int_t^T r(s)\,ds\right)$

$$V(t) = \widetilde{\mathbb{E}}\left[D(t)(S(T)-K)^+ \,\middle|\, \mathcal{F}_t\right]$$

need to find $\widetilde{\mathbb{P}}$.

Compute $d(D(t)S(t)) = D(t)\,dS(t) + S(t)\,dD(t) + dD(t)\,dS(t)$

$$\cancel{dD(t) = -r(t)D(t)\,dt} \quad dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$$

$$dD(t) = -r(t)D(t)\,dt.$$

$$\Rightarrow \quad d\,D(t)S(t) = D(t)S(t)\cdot\left[(\alpha - R(t))\,dt + \sigma\,dW(t)\right]$$

$$= D(t)S(t)\sigma\underbrace{\left[\frac{\alpha - R(t)}{\sigma}\,dt + dW(t)\right]}_{\text{want this to be a B.M } d\widetilde{W}(t)}$$

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$$

$\alpha > r$ want to convert it into a RN measure under which $S(t)$ has mean return r:

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t) \qquad \widetilde{W}(t) \text{ is a BM under } \widetilde{\mathbb{P}}$$

under RN measure, risky asset has mean return $r \iff$

$$d\,D(t)S(t) = \sigma S(t)\,d\widetilde{W}(t)$$

i.e. discounted stock price is a mtg.

Let $b(t) = \frac{\alpha - R(t)}{\sigma}$, define $Z(t) = \exp\left(-\int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b(s)^2 ds\right)$,

by Girsanov's Thm, $d\tilde{W}(t)$ is a B.M under $\tilde{\mathbb{P}}$ defined

by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z(T)$.

$$\mathbb{E}[X] = \int X \, d\mathbb{P}$$

$$\tilde{\mathbb{E}}[X] = \int X \, d\tilde{\mathbb{P}} = \int X \underbrace{\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}}_{Z(T)} \cdot d\mathbb{P} = \int X Z(T) \, d\mathbb{P}$$

$$= \mathbb{E}[Z(T) X]$$