FCE $\longrightarrow$ 75% Response Rate = Grades $\boxed{\text{Early}}$

Last time: Girsanov theorem.

$W \longrightarrow d$ -dim B.M.

$b = (b_1, b_2, \dots b_d)$

adapted process

$$d\tilde{W} = b(t)\,dt + dW$$

Define $Z(T) = \exp\Big(\underbrace{-\int_0^T b(s)\,dW(s)}_{-\int_0^T \sum_i b_i(s)\,dW_i(s)} - \frac{1}{2}\int_0^T |b(s)|^2 ds\Big)$

$dZ = -Z\,b(t)\cdot dW(t).$

$= -Z \sum_i b_i(t)\,dW_i(t)$

$d\tilde{P} = Z(T)\,dP$ $\qquad$ (T = some fixed time (maturity time)).

Then $\tilde{W}$ is a B.M. under $\tilde{P}$.

Risk Neutral measures:

Money market account $\rightarrow$ interest rate $R(t)$ (some adapted process).

Discount process: $D(t) = \exp\left(-\int_0^t R(s)\,ds\right)$

$$\boxed{\begin{aligned} &C(t) = \text{cash in M.M.} \\ &dC(t) = R(t)C(t)\,dt \end{aligned}}$$

$$dD(t) = -R(t)D(t)\,dt$$

(1 share of M.M. at time 0 is $\frac{1}{D(t)}$ at time t

$R(t) = r$ (constant). $D(t) =$ Discount factor at time $t = e^{-rt}$

$$= \exp\Big(-\int_0^t \underbrace{r}_{R(s)}\,ds\Big)$$

$$D(t)\,(\text{Cash at time } t) = \text{Cash at time } 0.$$

RNM: Market $\begin{cases} \to \text{M.M act} & \text{interest rate } R(t) \\ \to \text{Stock spot price} & S(t). \end{cases}$

~~$D(t)S(t)$~~

$$\boxed{\text{A Risk neutral measure is a measure } \tilde{P} \text{ under which } D(t)S(t) \text{ is a mg.}}$$

Remark: Fundamental theorems of asset pricing.

① Existence of a RNM $\Longleftrightarrow$ no arbitrage.

② Uniqueness of a RNM $\Longleftrightarrow$ (no arbitrage & every security can be hedged).

① Find RNM:

Compute $d\left(D(t)S(t)\right)$:

Assume $dS = \alpha(t) S\, dt + \sigma(t) S\, dW$

$$d\left(D(t)S(t)\right) = D(t)\,dS(t) + S(t)\,dD(t) + d\underbrace{[D(t), S(t)]}_{0}$$

$$= D(t)\left(\alpha S\, dt + \sigma S\, dW\right) - R(t)S(t)D(t)\, dt.$$

$$= \sigma(t) D(t) S(t) \Big(\underbrace{\frac{\alpha(t) - R(t)}{\sigma(t)}}_{\theta(t)} dt + dW \Big).$$

$$\boxed{\text{Let } \theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)} \quad (\text{market price of risk}).}$$

Girsanov & make $\theta(t)\,dt + dW = d\tilde{W}$ a BM under $\tilde{P}$.

$$d\tilde{W} = \theta(t)\,dt + dW.$$

$$Z(T) = \exp\left(-\int_0^T \theta(s)\,dW(s) - \frac{1}{2}\int_0^T \theta(s)^2\,ds\right).$$

Define $d\tilde{P} = Z(T)\,dP$

<u>Prop:</u> Under $\tilde{P}$, DS is a mg.

(the process $D(t)S(t)$).

(Note $d(DS) = \sigma(t)D(t)S(t)\,d\tilde{W}$).

Theorem: Risk Neutral Pricing formula. (IOU Justification)

Let $V(T)$ be an $\mathcal{F}_T$ meas RV.

$\left(V(T) \longrightarrow \text{payoff of some security. E.g. } V(T) = (S(T)-K)^+ \right)$,

The arbitrage free price of a derivative security with payoff $V(T)$ at time $t \leq T$ is given by.

$$\boxed{V(t) = \underset{\substack{\uparrow \\ \text{Expected value} \\ \text{under the RNM}}}{\tilde{E}} \left(V(T) \exp\left(-\int_t^T R(s)\,ds \right) \,\Big|\, \mathcal{F}_t \right),}$$

Remarks.

Compute dS in terms of $d\tilde{W}$.

$$dS = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)$$

$$= \alpha S\,dt + \sigma S\left(d\tilde{W} - \theta(t)\,dt\right).$$

$$= \alpha S\,dt + \sigma S\,d\tilde{W} - \sigma S\left(\frac{\alpha - R}{\sigma}\right)dt$$

$$= \underbrace{+R(t)S(t)\,dt}_{\text{mean return rate} \;=\; \text{interest rate!}} + \sigma(t)S(t)\,d\tilde{W}$$

B.M. under $\tilde{P}$.

Proof: Let $X(t)$ = wealth of an investor who holds $\Delta(t)$ shares of stock & rest cash.

If there is no external cash flow (self financing), then $D(t)X(t)$ is a mg under the RNM $\tilde{P}$.

Pf: $d\big(D(t)X(t)\big) \leftarrow$ compute.

$$dX(t) = \Delta(t)\,dS(t) + R(t)\big(X(t) - \Delta(t)S(t)\big)\,dt$$

$$\Rightarrow d(DX) = D\,dX + X\,dD + 0$$

$$= D\big(\Delta\,dS + R(X - \Delta S)\,dt\big) - RXD\,dt$$

$$= D\Delta\big(RS\,dt + \sigma S\,d\tilde{W}\big) - R\Delta SD\,dt \;\cancel{= \sigma S\,d\tilde{W}}\; \text{~~mg~~!!}$$

$$= D\Delta\sigma S\,d\tilde{W} \quad \text{mg} \quad //$$

Proof of RNPF: Security Payoff $V(T)$.

NTS: AFP at time $t = \tilde{E}\left(V(T)\exp\left(-\int_t^T R(s)\,ds\right) \,\middle|\, \mathcal{F}_t\right)$.

Let $X(t)$ = wealth of the R. Portfolio.

$\Rightarrow$ $D(t)X(t)$ is a mg UNDER $\tilde{P}$ (R.N.M.).

$V(t)$ = AFP at time t of the security $= X(t)$

$$= \frac{1}{D(t)} D(t)X(t) \overset{\text{prop}}{=} \frac{1}{D(t)} \tilde{E}\left(D(T)X(T) \,\middle|\, \mathcal{F}_t\right).$$

$$= \frac{1}{D(t)} \tilde{E}\left(D(T)V(T) \,\middle|\, \mathcal{F}_t\right) = \tilde{E}\left(\frac{D(T)}{D(t)} V(T) \,\middle|\, \mathcal{F}_t\right)$$

//.

Derive Black Scholes formula using RNM.

$S = GBM(\alpha, \sigma)$ α & σ are now constants.

Intrest rate r (constant, not random).

$D(t) = e^{-rt}$. $c(t, x) =$ price of European call strike K maturity T.

$$\text{RNP} \Rightarrow c(t, S(t)) = \tilde{E}\left(e^{-r(T-t)} (S(T) - K)^+ \,\middle|\, \mathcal{F}_t \right)$$

Compute RHS: Bad way $\longrightarrow$ Have formula for $\tilde{P}$ & $\tilde{E}$ in terms of Z & substitute.

Better way: Know $dS = r\,dt + \sigma\,d\tilde{W}$

$\Rightarrow$ Under RNM, $\tilde{P}$, S is a GBM(r, σ)

$$\Rightarrow S(T) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}(T)\right).$$

$$\Rightarrow S(t) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}(t)\right).$$

$$\Rightarrow S(T) = S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}(T) - \tilde{W}(t)\right)\right).$$

$\boxed{\tau = T - t.}$

$$\Rightarrow c(t, S(t)) = \tilde{E}\left(e^{-r\tau}\left(\underbrace{S(t)}_{\mathcal{F}_t \text{ meas.}}\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sqrt{\tau}\,\sigma\,\underbrace{\left(\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{\tau}}\right)}_{\substack{\sim N(0,1) \\ \& \text{ indep of } \mathcal{F}_t}}\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right).$$

Indep Lemma

$$\Rightarrow c(t, \underbrace{S(t)}_{x}) = \int_{y \in \mathbb{R}} e^{-r\tau} \Big[\underbrace{S(t)}_{x} \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, y \Big) - K \Big]^{+} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} .$$

$(*)$

Put $S(t) = x$. Find y where ≥ 0

$$x \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, y \Big) \geq K$$

$$\Leftrightarrow \big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, y \geq \ln\Big(\frac{K}{x}\Big) = -\ln\Big(\frac{x}{K}\Big).$$

$$\Leftrightarrow y \geq -\frac{1}{\sigma\sqrt{\tau}} \Big(\ln\Big(\frac{x}{K}\Big) + \big(r - \tfrac{\sigma^2}{2}\big)\tau \Big) = -d_{-}$$

$$\circledast \Rightarrow c(t,x) = \int_{y=-d_-}^{\infty} e^{-r\tau}\left(x e^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}\, y} - K\right) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}.$$

$$= \int_{-d_-}^{\infty} x e^{-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}\, y - \frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} - K e^{-r\tau} \int_{-d_-}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

$$= x \int_{-d_-}^{\infty} e^{-\frac{1}{2}\left(\sigma^2\tau - 2\sigma\sqrt{\tau}\, y + y^2\right)} \frac{dy}{\sqrt{2\pi}} - K e^{-r\tau} \int_{-\infty}^{d_-} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

$$= x \int_{-d_-}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2} \frac{dy}{\sqrt{2\pi}} - K e^{-r\tau} N(d_-)$$

Put $z = y - \sigma\sqrt{\tau}$

$$= x \int_{-d_- - \sigma\sqrt{\tau}}^{\infty} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} - Ke^{-r\tau} N(d_-)$$

$$= x N(d_- + \sigma\sqrt{\tau}) - Ke^{-r\tau} N(d_-).$$

Note $d_- + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right) + \sigma\sqrt{\tau}$

$$= \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right) = d_+.$$

$$\boxed{\Rightarrow c(t,x) = x N(d_+) - Ke^{-r\tau} N(d_-)}$$

//