Today: - Multidimensional Ito

- Lèvy's Theorem.

- Change of measure / Girsanov's.

Multivariate Ito.

$X_t = (X_t^1, \dots, X_t^d) \to d$ dimensional Ito process

twice cont. differentiable

$f(t, x_1, \dots, x_d) = f(t, \bar{x})$

once cont. differentiable.

i.e. $\partial_{ij} f$ exists for all $i, j \in \{1, \dots, d\}$.

Then

$$df(t, X_t) = \partial_t f(t, X_t)\,dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\,dX_t^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_{ij} f(t, X_t)\,d[X^i, X^j]_t$$

Ex
$$\begin{cases} dX_t = \mu_1 X_t\,dt + \sigma_1 X_t\,dW_t^1 \\ dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^2 \end{cases}$$

$$d[W^1, W^2]_t = \rho\,dt \quad (\rho \in [-1,1] \text{ constant}).$$

Define $Z_t = \dfrac{X_t}{Y_t}$

(i) Find dZ_t.

$$f(x,y) = \frac{x}{y}$$

$$\partial_x f = \frac{1}{y} \qquad \partial_y f = \frac{-x}{y^2}$$

$$\partial_{yx} f = \partial_{xy} f = -\frac{1}{y^2} \qquad \partial_{xx} f = 0 \qquad \partial_{yy} f = \frac{2x}{y^3}$$

$$d f(X_t, Y_t) = \frac{dX_t}{Y_t} + \frac{-X_t}{Y_t}\frac{dY_t}{Y_t} - \frac{1}{Y_t^2}\, d[X,Y]_t + \frac{1}{2}\frac{2X_t}{Y_t}\frac{d[Y,Y]_t}{Y_t^2}$$

$$= \frac{X_t}{Y_t}\left(\mu_1 dt + \sigma_1 dW^1_t\right) - \frac{X_t}{Y_t}\left(\mu_2 dt + \sigma_2 dW^2_t\right)$$

$$- \frac{1}{Y_t^{\cancel{2}}} \sigma_1 \sigma_2 X_t \cancel{Y_t} \underbrace{d[W^1, W^2]_t}_{\rho dt} + \frac{X_t}{Y_t} \sigma_2^2 dt$$

$$= \underbrace{\frac{X_t}{Y_t}}_{Z_t}\left[\left(\mu_1 - \mu_2 - \rho\sigma_1\sigma_2 + \sigma_2^2\right) dt + \sigma_1 dW^1_t - \sigma_2 dW^2_t\right]$$

(ii) Compute $d[Z,Z]_t$.

$$d[Z,Z]_t = \left[Z_t\left((\cdots)dt + \sigma_1 dW^1_t - \sigma_2 dW^2_t\right), Z_t\left((\cdots)dt + \sigma_1 dW^1_t - \sigma_2 dW^2_t\right)\right]$$

$$= \left[Z_t\left(\sigma_1 dW^1_t - \sigma_2 dW^2_t\right), Z_t\left(\sigma_1 dW^1_t - \sigma_2 dW^2_t\right)\right].$$

We will use bilinearity:

aside: T, U, V & ~~Z~~ processes.

$$d[T+U, V]_t = d[T,V]_t + d[U,V]_t$$

$$d[Z,Z]_t = [Z_t\sigma_1 dW_t^1, Z_t(\sigma_1 dW_t^1 - \sigma_2 dW_t^2)] - [Z_t\sigma_2 dW_t^2, Z_t(\sigma_1 dW_t^1 - \sigma_2 dW_t^2)].$$

$$= [Z_t\sigma_1 dW_t^1, Z_t\sigma_1 dW_t^1] - [Z_t\sigma_1 dW_t^1, Z_t\sigma_2 dW_t^2] - [Z_t\sigma_2 dW_t^2, Z_t\sigma_1 dW_t^1] + [Z_t\sigma_2 dW_t^2, Z_t\sigma_2 dW_t^2].$$

$$= Z_t^2\sigma_1^2 \cancel{\sigma_2}\, dt - 2Z_t^2\rho\sigma_1\sigma_2\, dt + Z_t^2\sigma_2^2\, dt$$

$$= Z_t^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)\, dt.$$

if $\rho=1$: $Z_t^2(\sigma_1-\sigma_2)^2$ so always positive!

If $Y_t = b_t\,dt + \sum_{i=1}^{d} \sigma_i\, dX_t^i$

where X_t^i Ito processes.

$$d[Y_t, Y_t] = d\left[\sum_{i=1}^{d} \sigma_i \cancel{d} X^i, \sum_{j=1}^{d} \sigma_j \cancel{d} X^j\right]_t$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j\, d[X_t^i, X_t^j]_t$$

Lévy's T

Lévy's Theorem:

Let $M = (M^1, \dots, M^d)$ be a continuous d-dimensional martingale with $M_0 = (M_0^1, M_0^2, \dots, M_0^d) = \vec{0}$

If $[M_i, M_j]_t = t\,\mathbb{1}\{i=j\}$.

then M is a d-dimensional Brownian Motion.

Pf $M_0 = 0$ ✓

Continuous paths ✓

Remains to check (i) $M_t - M_s \sim \mathcal{N}(0, (t-s)I_d)$. $s \le t$.

(ii) $M_t - M_s \perp\!\!\!\perp \mathcal{F}_s$.

Use M.G.F's :

If $X = (X_1, \dots, X_d)$ is a d-dimensional RV.

then $\quad M_X(\lambda) = \mathbb{E}[e^{X \cdot \lambda}] = \mathbb{E}[e^{\lambda_1 X_1 + \dots + \lambda_d X_d}]$,

(λ is a vector here).

we will compute $\mathbb{E}[e^{(M_t - M_s)\cdot \lambda} \mid \mathcal{F}_s]$.

$$f(x_1, \dots, x_d) = e^{\sum_{i=1}^{d} x_i \lambda_i}$$

$$\partial_i f = \lambda_i f$$

$$\partial_{ij} f = \lambda_i \lambda_j f \qquad (\partial_{ii} f = \lambda_i^2 f).$$

$$d f(M_t) = \sum_{i=1}^{d} \lambda_i f(M_t)\, dM_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \lambda_i \lambda_j f(M_t)\, d[M^i, M^j]_t$$

$$\overset{\parallel}{d(e^{M_t \cdot \lambda})} = \sum_{i=1}^{d} \lambda_i f(M_t)\, dM_t^i + \frac{1}{2} \underbrace{\sum_{i=1}^{d} \lambda_i^2}_{\|\lambda\|^2} f(M_t)\, dt$$

$$e^{M_t \cdot \lambda} = e^{M_s \cdot \lambda} + \sum_{i=1}^{d} \int_s^t \lambda_i f(M_u)\, dM_u^i + \int_s^t \frac{1}{2} \|\lambda\|^2 f(M_u)\, du.$$

$$\mathbb{E}\left[e^{M_t \cdot \lambda} - e^{M_s \cdot \lambda} \middle| \mathcal{F}_s\right] = 0 + \int_s^t \frac{1}{2}\|\lambda\|^2 \, \mathbb{E}\left[e^{M_u \cdot \lambda} \middle| \mathcal{F}_s\right] du.$$

divide both sides by $e^{M_s \cdot \lambda}$

$$\mathbb{E}\left[e^{(M_t - M_s)\cdot\lambda} - 1 \middle| \mathcal{F}_s\right] = \frac{1}{2}\|\lambda\|^2 \int_s^t \mathbb{E}\left[e^{(M_u - M_s)\cdot\lambda} \middle| \mathcal{F}_s\right] du.$$

define $g(t) := \mathbb{E}\left[e^{(M_t - M_s)\cdot\lambda} \middle| \mathcal{F}_s\right]$. $t \geq s$:

$$g(t) = 1 + \frac{1}{2}\|\lambda\|^2 \int_s^t g(u)\, du.$$

$$\begin{cases} g'(t) = \frac{1}{2}\|\lambda\|^2 g(t) \\ g(s) = 1 \end{cases}$$

$$\Rightarrow g(t) = e^{\frac{1}{2}\|\lambda\|^2 (t-s)}.$$

$$\mathbb{E}\left[e^{(M_t - M_s)\cdot\lambda} \mid \mathcal{F}_s\right] = e^{\frac{1}{2}\|\lambda\|^2 (t-s)} \quad (*) .$$

↳ MGF of $N(0, (t-s) I_d)$

↳ will show why.

$$\Rightarrow \quad M_t - M_s \sim N(0, (t-s) I_d) .$$

Notice in $(*)$ RHS is non-random. With some work ~~work~~ one can show that $M_t - M_s \perp\!\!\!\perp \mathcal{F}_s$.

<u>Warning</u> In general if I have a RV X and a σ-algebra $\mathcal{F}$ with

$\mathbb{E}[X \mid \mathcal{F}] =$ constant this does <u>not</u> imply

$X \perp\!\!\!\perp \mathcal{F}$

If $(X_1, \dots, X_d) \sim N(0, (t-s) I)$

in particular X_i is indep of X_j when $i \neq j$.

$$\mu_X(\lambda) = \mathbb{E}\left[e^{\sum_{i=1}^d \lambda_i X_i}\right] = \mathbb{E}\left[\prod_{i=1}^d e^{\lambda_i X_i}\right] = \prod_{i=1}^d \mathbb{E}[e^{\lambda_i X_i}].$$

$$= \prod_{i=1}^d e^{\frac{1}{2}\lambda_i^2 (t-s)} = e^{\frac{1}{2}\sum_{i=1}^d \lambda_i^2 (t-s)} = e^{\frac{1}{2}\|\lambda\|^2 (t-s)}$$

$\square$

<u>Ex</u>: Take $p \in [\cancel{\tfrac{1}{2}}\,0, 1]$ and independent BM's B, W

$$M_t = \sqrt{1-p}\, W_t + \sqrt{p}\, B_t.$$

Actually M_t is a (1-dim) BM!

<u>pf</u> $M_0 = 0$ ✓ continuous ✓

just need to verify that $[M,M]_t = t$

$$[M,M]_t = [\sqrt{1-p}\, W_t, \sqrt{1-p}\, W_t] + 2[\sqrt{p}\, B_t, \sqrt{1-p}\, W_t] + [\sqrt{p}\, B_t, \sqrt{p}\, B_t]$$

$$= (1-p)t + 0 + pt$$

$$= t$$

So By Lévy M_t is a BM. $\square$.

If $\mathbb{P}$ is a prob measure. and Z RV s.t.

$Z > 0$ ~~a.s.~~ and $\mathbb{E}[Z] = 1$ then we can

define a new prob measure $\tilde{\mathbb{P}}$ given

$$\tilde{\mathbb{P}}(A) = \mathbb{E}[Z \mathbb{1}_A].$$

↳ under $\mathbb{P}$.

or we write $d\tilde{\mathbb{P}} = Z\, d\mathbb{P}$ (means same as line above).

and $\tilde{\mathbb{P}}$ is equivalent $\mathbb{P}$.

i.e. $\mathbb{P}(A) = 0 \iff \tilde{\mathbb{P}}(A) = 0$ and $\tilde{\mathbb{P}}(A) = 1 \iff \mathbb{P}(A) = 1$

" $\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree on what is possible or not possible, but may disagree on how likely things are "

<u>Ex</u> suppose $X \sim \mathcal{N}(0,1)$ under $\mathbb{P}$.

define $d\tilde{\mathbb{P}} = \underbrace{e^{\alpha X + \beta}}_{Z} d\mathbb{P}$.

α is given to you.

Goal: find $\beta, \gamma \in \mathbb{R}$ s.t. $X + \gamma \sim \mathcal{N}(0,1)$ under $\tilde{\mathbb{P}}$.

Sol'n $\quad Z > 0$ ✓

$$\mathbb{E}[Z] = e^{\beta}\,\mathbb{E}[e^{\alpha X}] = e^{\beta + \frac{1}{2}\alpha^2} \overset{\text{set}}{=} 1$$

$$\rightarrow \boxed{\beta = -\frac{1}{2}\alpha^2}.$$

Now we need $X_T + \gamma \sim N(0,1)$ under $\tilde{\mathbb{P}}$.

$$\tilde{\mathbb{P}}(X + \gamma \le x) = \tilde{\mathbb{P}}(X \le x - \gamma) = \int_{-\infty}^{x-\gamma} Z\, f_X(y)\,dy.$$

↳ standard normal PDF.

$$\int \mathbb{1}\{X \le x - \gamma\}\, d\tilde{\mathbb{P}}$$

$$\|$$

$$\int \mathbb{1}\{X \le x - \gamma\}\, Z\, d\mathbb{P}.$$

$$= \int_{-\infty}^{\infty} \mathbb{1}\{Y \le x - \gamma\}\, Z\, f_X(y)\,dy.$$

$$= \int_{-\infty}^{x-\gamma} e^{\alpha y - \frac{1}{2}\alpha^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy.$$

$$= \int_{-\infty}^{x-\gamma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\alpha)^2}\, dy.$$

set $\cancel{z}\ w = y - \alpha.$

$$\Rightarrow \int_{-\infty}^{x-\gamma-\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\, dw.$$

$$= N(x-\gamma-\alpha).$$

Need ~~//~~ this to equal $N(x)$.

Happens $\Leftrightarrow$ $\gamma = -\alpha.$

$\therefore$ If $Z = e^{\alpha X - \frac{\alpha^2}{2}}$ then $X - \alpha$ is $N(0,1)$ under $\tilde{\mathbb{P}}$.

~~$\Leftrightarrow$~~ $X \sim N(0,1)$ under $\mathbb{P}$.

and $X \sim$ ~~$\cdot$~~ $N(\alpha, 1)$ under $\tilde{\mathbb{P}}$.

~~$\Leftrightarrow$~~ i.e we changed the mean, but not the variance.

1-D Girsanov:

let $\tilde{W}_t = W_t + \int_0^t b(s)\, ds.$

$$Z_t = e^{-\int_0^t b(s)\, dW_s - \frac{1}{2}\int_0^t b_s^2\, ds.}$$

If $\mathbb{E}[Z_t] = 1$ then

$\tilde{W}_t$ is a BM under $\tilde{\mathbb{P}}$ where $d\tilde{\mathbb{P}} = Z_t\, d\mathbb{P}$.

this looks similar to what we just did!

if " $\alpha = -\int_0^t b(s)\, dW_s$, $\beta = -\frac{1}{2}[\alpha, \alpha]$.

Notice Girsanov changes the drift, similar to how the example changed the mean.

But the volatility is not affected.