Last time: $X = (X_1, \dots X_d)$

$f = f(t, x_1, \cdots x_d)$ $\quad f = f(t, x)$ [with $x = (x_1, x_2, \dots x_d)$]

Itô: $$d f(t,X) = \partial_t f \, dt + \sum_{i=1}^{d} \partial_i f \, dX_i + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f \, d[X_i, X_j](t)$$

Multi dim BM: $W = (W_1, \dots W_d)$ is a d-dim std B.M. if

① Each W_i is a std 1D B.M.

& ② for $i \neq j$ W_i is ind of W_j

Compute: $$d[W_i, W_j](t) = \begin{cases} dt & \text{if } i = j \\ 0\,dt & \text{if } i \neq j \end{cases}$$

Recitation tomorrow (Q7.5).

Levy: M is a d-dim cts mg. If $d[M_i, M_j] = \begin{cases} dt & i = j \\ 0\,dt & i \neq j \end{cases} \Rightarrow M$ is a B.M.

$M = (M_1, M_2, \dots M_d)$

Eg 1: Say $f = f(t, x_1, \dots x_d)$. & $W = d$ dim B.M.

Compute $d\left(f(t, W(t)\right) \overset{\text{Itô}}{=\!=} \partial_t f(t, W(t))\, dt + \sum_{i=1}^{d} \partial_i f(t, W(t))\, dW_i(t)$

$$+ \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f(t, W(t))\, d[W_i, W_j](t).$$

$$= \partial_t f\, dt + \sum \partial_i f\, dW_i(t) + \frac{1}{2} \sum_{i=1}^{d} \partial_i^2 f(t, W(t))\, dt$$

Note $\sum_{i=1}^{d} \partial_i^2 f = \partial_1^2 f + \partial_2^2 f + \cdots \partial_d^2 f =$ Laplacian of $f = \Delta f$

$\Rightarrow d f(t, W(t)) = \left(\partial_t f + \frac{1}{2} \Delta f\right) dt + \sum \partial_i f\, dW_i(t)$.

Eg 2: Choose $d=2$. $\quad W=(W_1, W_2)$ $\quad$ (2D B.M.).

$$f(t,x) = f(t, x_1, x_2) = \ln(|x|) = \ln\sqrt{x_1^2+x_2^2}.$$

$$= \frac{1}{2}\ln(x_1^2+x_2^2).$$

$$\partial_t f = 0 \qquad \partial_1 f = \frac{1}{2}\,\frac{1}{|x|^2}\,2x_1 = \frac{x_1}{|x|^2}$$

$$\partial_2 f = \frac{x_2}{|x|^2}$$

$$\cancel{\partial_i\partial_j f} \quad \partial_1^2 f + \partial_2^2 f = \Delta f = \text{You check} = 0$$

$$\Rightarrow d\ln|W(t)| = \sum_{i=1}^{2} \frac{W_i(t)}{|W(t)|^2}\, dW_i + \frac{1}{2}\underbrace{\Delta f}_{0}\, dt$$

$$\Rightarrow d\left(\ln|W(t)|\right) = \underbrace{\frac{W_1(t)\, dW_1(t)}{|W(t)|^2}}_{mg} + \underbrace{\frac{W_2(t)\, dW_2(t)}{|W(t)|^2}}_{mg}.$$

$$E \ln|W(t)| = \int_{\mathbb{R}^2} \ln|x|\, e^{-|x|^2/2t} \frac{dx_1\, dx_2}{2\pi t} \cancel{=0} \xrightarrow{t\to\infty} +\infty$$

Claim: RHS is a "local martingale" (& not a mg).

~~Fe~~ For $\int_0^t \sigma(s)\, dW(s)$ to be a mg need

σ to be adapted. & $E\int_0^t \sigma(s)^2 ds < \infty$.

Risk Neutral measures:

<u>Motivation:</u> ~~Say~~ Financial market. Interest rate r.

$X(t)$ = wealth of the R. Pf of a security with payoff $V(T)$ at time T.

$\boxed{\text{Suppose}}$ $\underbrace{e^{-rt} X(t)}$ is a mg.

↑ Discounted wealth.

Then can compute $X(t)$ in terms of $V(T)$!!

Note $E(V(T) \mid \mathcal{F}_t) = E(X(T) \mid \mathcal{F}_t) = e^{+rT} E(e^{-rT} X(T) \mid \mathcal{F}_t)$

$$= e^{+rT} e^{-rt} X(t)$$

$$\boxed{\Rightarrow X(t) = e^{-r(T-t)} E\left(V(T) \mid \mathcal{F}_t\right)}$$

If Disc wealth is a mg then price securities by

Usually disc wealth is not a mg.

Risk Neutral Measure: Constut a new measure under which the discounted wealth process is a mg

① Equivalent measures: $P \longrightarrow$ prob measure.

$\tilde{P}$ a new prob measure (Financial app: Risk Neutral measure).

P & $\tilde{P}$ are equivalent if ~~whenever~~ $P(A)=0 \Longleftrightarrow \tilde{P}(A)=0$.

Eg: Z be a R.V. Assume $Z>0$ almost surely.

& $EZ = 1$.

Define a new measure $\tilde{P}$ by

$$\tilde{P}(A) = \int_A Z \, dP = E(\mathbb{1}_A Z) \in [0,1].$$

Note $P(A) = 0 \Longleftrightarrow \tilde{P}(A)=0$ (i.e. $\tilde{P}$ & P are equiv).

Notation: If $\tilde{P}(A) = \int_A Z \, dP$ write $d\tilde{P} = Z \, dP$.

Reason: Can check $\int_\Omega X \, d\tilde{P} = \int_\Omega X Z \, dP$

$\tilde{E} \longrightarrow$ Expected value wrt the new measure $\tilde{P}$.

$\tilde{E}(X \mid \mathcal{F}) \longrightarrow$ cond exp of X given the σ alg $\mathcal{F}$ under the new measure $\tilde{P}$.

Thm (Cameron - Martin - Girsanov thm).

$$b(t) = \left(b_1(t),\ b_2(t),\ \ldots\ \cancel{d}\, b_d(t)\right) \leftarrow d \text{ dim adapted process.}$$

$$W(t) = \left(W_1(t),\ \ldots\ W_d(t)\right) \longleftarrow d \text{ dim B.M..}$$

Let $\tilde{W}(t) = W(t) + \int_0^t b(s)\, ds.$

Let $$Z(T) = \exp\left(-\int_0^T b(s)\cdot dW(s) - \frac{1}{2}\int_0^T |b(s)|^2\, ds\right).$$

$$= \exp\left(-\int_0^T \sum_{i=1}^d b_i(s)\, dW_i(s) - \frac{1}{2}\int_0^T \sum b_i(s)^2\, ds\right).$$

$\boxed{\text{If } Z \text{ is a mg,}}$ then define $d\tilde{P} = Z(T)\, dP.$

Then $\tilde{W}$ is a B.M. under $\tilde{P}$ (up to time T).

Note: Compute dZ.

Let $M(t) = -\int_0^t b(s) \cdot dW(s) = -\int_0^t \sum b_i(s)\, dW_i(s)$

$$\Rightarrow [M,M](t) = \sum_{i=1}^{d} \int_0^t b_i(s)^2\, ds = \cancel{\sum_{i=1}^{d}} \int_0^t |b(s)|^2\, ds.$$

$$\Rightarrow Z(t) = \exp\left(M(t) - \frac{1}{2} [M,M](t) \right).$$

Itô: Let $f(t,x) = \exp\left(x - \frac{1}{2}[M,M](t)\right)$.

$$\partial_t f = -\frac{1}{2} f\, \partial_t [M,M]. \qquad \partial_x f = f \;\; \& \;\; \partial_x^2 f = f.$$

$$dZ = \partial_t f\, dt + \partial_x f\, dM + \frac{1}{2} \partial_x^2 f\, d[M,M].$$

$$= -\frac{1}{2} Z \, \cancel{d[M,M]} + Z\,dM + \frac{1}{2} Z \, \cancel{d[M,M]}$$

$$= Z\,dM = Z(t)\left(-b(t)\cdot dW(t)\right).$$

$$dZ(t) = -\sum_{i=1}^{d} Z\, b_i(t)\, dW_i(t) \longleftarrow \text{looks like a mg!}$$

(could be a LOCAL mg).

Need to assume Z is a mg in the thm.

Note: for $\tilde{P}$ to be a prob measure we need

$$E\, Z(T) = 1.$$

Since Z is a mg, $E\,Z(T) = E\,Z(0) = 1.$

Pf of Girsanov:

Lemma Let $0 \leq s \leq t$. $X \to \mathcal{F}_t$ meas R.V.

Then $\tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t) X \mid \mathcal{F}_s)$. ---- (*)

Cond exp wrt $\nearrow$ new meas $\tilde{P}$ Cond exp wrt P.

Pf: Recall: If Y is any R.V.

then $\int_A \tilde{E}(Y \mid \mathcal{F}_s)\, d\tilde{P} = \int_A Y \, d\tilde{P}$ for every $A \in \mathcal{F}_s$.

Check (*): Let $A \in \mathcal{F}_s$.

$$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} \overset{①}{=} \int_A Z(T)\, \tilde{E}(X \mid \mathcal{F}_s)\, dP$$

$$= \int_A E\big(Z(T)\, \tilde{E}(X \mid \mathcal{F}_s) \mid \mathcal{F}_s \big)\, dP \quad (\because A \in \mathcal{F}_s).$$

$$= \int_A \tilde{E}(X \mid \mathcal{F}_s)\, \underbrace{E(Z(T) \mid \mathcal{F}_s)}\, dP$$

$$= \int_A \tilde{E}(X \mid \mathcal{F}_s) \quad Z(s) \quad dP \quad \cdots \quad (**)$$

②

$$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} \overset{②}{=\!=} \int_A X \, d\tilde{P} = \int_A Z(T) X \, dP$$

$$= \int_A E\big(Z(T)X \mid \mathcal{F}_s\big)\, dP$$

$$= \int_A E\Big(E\big(Z(T)\, X \mid \mathcal{F}_t\big) \mid \mathcal{F}_s\Big)\, dP$$

$$= \int_A E\big(Z(t)X \mid \mathcal{F}_s\big)\, dP \quad \cdots\cdots \; ***$$

$$\Rightarrow Z(s)\, \tilde{E}(X \mid \mathcal{F}_s) = E\big(Z(t) X \mid \mathcal{F}_s\big).$$

//

Lemma 2: An adapted process M is a mg under $\tilde{P}$

$\iff$ ZM is a mg under P.

Pf: Say MZ is a mg under P.

NTS M is a mg under $\tilde{P}$.

Compute $\tilde{E}(M(t) \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t) M(t) \mid \mathcal{F}_s)$

$$= \frac{1}{\cancel{Z(s)}} \cancel{Z(s)} M(s) = M(s)$$

//.

Pf of Girsanov. $d\tilde{W} = b(t)\,dt + dW(t)$.

NTS $\tilde{W}$ is a ~~ma~~ B.M. under $\tilde{P}$.

Pf: Levy. ① Check $\tilde{W}$ is a ~~m~~ cts mg under $\tilde{P}$

② Check $d[\tilde{W}_i, \tilde{W}_j] = \begin{cases} 0\,dt & i \neq j \\ dt & i = j \end{cases}$.

Note ② is immediately true! $(d[\tilde{W}_i, \tilde{W}_j] = d[W_i, W_j]$.

For ①. Only need to check $Z\tilde{W}$ is a mg under P.

$$d(Z\tilde{W}_j) = Z\,d\tilde{W}_j + \tilde{W}_j\,dZ + d[Z, \tilde{W}_j].$$

$$= \underbrace{\tilde{W}_j\,dZ}_{mg} + \cancel{Z b_j\,dt} + \underbrace{Z\,dW_j}_{mg} - \cancel{Z b_j\,dt}$$

QED.