Levy's characterization Thm

If $M(t)$ is an adapted process w.r.t. filtration $\mathcal{F}_t$, if $M(t)$ is a continuous mtg and $[M,M](t)=t$, then $M(t)$ is a B.M. w.r.t. $\mathcal{F}_t$.

Let $W_1(t)$ $W_2(t)$ be two independent B.Ms, find an adapted process σ s.t. $B(t)$ defined by:

$$B(t) = \int_0^t \frac{1}{1+W_1(s)^2}\, dW_1(s) + \int_0^t \sigma(s, W_1(s), W_2(s))\, dW_2(s)$$

is a B.M.

It is clear that $B(t)$ is a continuous mtg. we only need $d[B,B](t) = dt$ to conclude that $B(t)$ is a B.M.

(from Levy's characterization Thm).

$$dB(t) = \frac{1}{1+W_1(t)^2} dW_1(t) + \sigma(t, W_1(t), W_2(t))\, dW_2(t)$$

$$d[B,B](t) = \left(dB(t)\right)^2 = \left(\frac{1}{1+W_1(t)^2}\right)^2 \underbrace{\left(dW_1(t)\right)^2}_{dt} + \sigma^2 \underbrace{\left(dW_2(t)\right)^2}_{dt}$$

$$+ \frac{2}{1+W_1(t)} \sigma . \underbrace{dW_1(t)\, dW_2(t)}_{0}$$

$$= \underbrace{\left[\left(\frac{1}{1+W_1(t)^2}\right)^2 + \sigma^2\right] dt}_{\text{want this to be } dt}.$$

$$\Rightarrow \sigma^2(t, W_1(t), W_2(t)) = 1 - \left(\frac{1}{1+W_1(t)^2}\right)^2$$

$$\Rightarrow \sigma(t, W_1(t), W_2(t)) = \sqrt{1 - \left(\frac{1}{1+W_1(t)^2}\right)^2}$$

Let $W = (W_1(t), W_2(t))$ be a standard 2-d B.M. define

$$X(t) = t + \int_0^t \mathbb{1}_{\{W_1(s) > W_2(s)\}} dW_1(s) + \int_0^t \mathbb{1}_{\{W_1(s) \le W_2(s)\}} dW_2(s)$$

Compute $[X,X](t)$ and $\mathbb{E} e^{7X(t)}$

$$dX(t) = dt + \mathbb{1}_{\{W_1(t) > W_2(t)\}} dW_1(t) + \mathbb{1}_{\{W_1(t) \le W_2(t)\}} dW_2(t)$$

$$d[X,X](t) = (dX(t))^2$$

$$= (dt)^2 + \left(\mathbb{1}_{\{W_1(t) > W_2(t)\}}\right)^2 (dW_1(t))^2 + \left(\mathbb{1}_{\{W_1(t) \le W_2(t)\}}\right)^2 (dW_2(t))^2$$

$$= 0 + \mathbb{1}_{\{W_1(t) > W_2(t)\}} dt + \mathbb{1}_{\{W_1(t) \le W_2(t)\}} dt.$$

$= dt$

$\Rightarrow [X,X](t) = t$

Note that $X(t)-t$ is a mtg with continuous path.

$[X(t)-t, X(t)-t](t) = [X,X](t) = t$. this is because

$d(X(t)-t) = dX(t) - dt$

$\Rightarrow \cancel{d(X(t)-t)^2 = d}\; d[X(t)-t, X(t)-t] = \left(d(X(t)-t)\right)^2$

$= (dX(t))^2 - 2\,dX(t)\,dt + (dt)^2.$

$= (dX(t))^2 = d[X,X](t)$

So by Levy's characterization Thm, $X(t)-t$ is a B.M.

$$\Rightarrow \quad \mathbb{E} e^{7X(t)} = \mathbb{E} e^{7(X(t)-t+t)} = \mathbb{E} e^{7t} \cdot e^{7(X(t)-t)}$$

$$= e^{7t}\, \mathbb{E} e^{7(X(t)-t))} \qquad X(t)-t \sim N(0,t)$$

$$\underline{\underline{m.g.f}} \quad e^{7t} e^{\frac{49t}{2}}$$

$$= e^{\frac{63t}{2}}$$

Product Rule:

For two processes $X(t)$, $Y(t)$:

$$d\, X(t)Y(t) = X(t)\,dY(t) + Y(t)\,dX(t) + \underbrace{dX(t)\,dY(t)}_{d[X,Y](t)}$$

This product rule can be derived from 2-d Ito's lemma with $f(t,x,y)=xy$, then calculate $df(t,X(t),Y(t))$ using 2-d Ito's lemma.

Compute $\mathbb{E}\Big(W(t) \underbrace{\int_0^t e^{3W(s)}\,dW(s)}_{:=M(t)} \Big)$

Let $M(t) := \int_0^t e^{3W(s)}\,dW(s)$, $X(t) = W(t)M(t)$. by product rule

$$
\begin{aligned}
dX(t) &= W(t)\,dM(t) + M(t)\,dW(t) + dW(t)\,dM(t) \\
&= W(t)\,e^{3W(t)}\,dW(t) + M(t)\,dW(t) + \underbrace{dW(t)\cdot e^{3W(t)}\,dW(t)}_{=e^{3W(t)}\,dt} \\
&= W(t)\,e^{3W(t)}\,dW(t) + M(t)\,dW(t) + e^{3W(t)}\,dt
\end{aligned}
$$

Integrate from 0 to t:

$$X(t) = \underbrace{\int_0^t W(s) e^{3W(s)} dW(s) + \int_0^t M(s) dW(s)}_{\text{is a mtg with expectation } 0} + \int_0^t e^{3W(s)} ds$$

$$\Rightarrow \quad \mathbb{E} X(t) = 0 + \mathbb{E} \int_0^t e^{3W(s)} ds$$

$$= \int_0^t \mathbb{E} e^{3W(s)} ds$$

$$\overset{\text{m.g.f}}{=} \int_0^t e^{\frac{9s}{2}} ds$$