Goal $\to$ ① Multi Dimensional Itô formula. (Today).

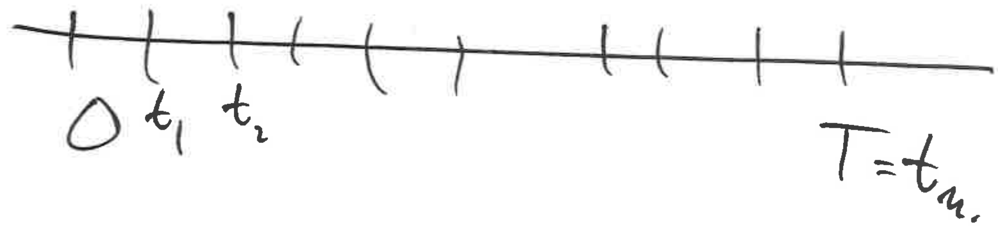
② Risk Neutral Measures. (R N Pricing Formula).

③ Consequence: Re-derive Black Sholes formula.

Joint Quadratic Variation:

$P = \{0 = t_0 < t_1 \dots t_n = T\}$.

QV: $X \to$ some process.



$$[X,X](T) = \lim_{\|P\| \to 0} \sum (\Delta_i X)^2$$

$$\|P\| = \max_i \; t_{i+1} - t_i \quad \& \quad \Delta_i X = X(t_{i+1}) - X(t_i)$$

Worked because for mg's, $(\Delta_i X)^2 \approx O(t_{i+1} - t_i)$

$\Rightarrow \quad (\Delta_i X) \approx O\left((t_{i+1}-t_i)^{1/2}\right).$

Given two processes X & Y: $\quad (\Delta_i X)(\Delta_i Y) \approx O(t_{i+1}-t_i)$

Joint QV: X & Y two processes.

The Joint Quadratic Variation of X & Y is

$$[X,Y](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \underbrace{(\Delta_i X)}_{[X(t_{i+1})-X(t_i)]} \underbrace{(\Delta_i Y)}_{[Y(t_{i+1})-Y(t_i)]}.$$

($P \to$ partition of $[0,T]$).

$[X,Y]$ is an adapted process.

Finance Notation.

$$d[X,Y] = dX\,dY$$

$$d[X,X] = dX\,dX$$

Recall: $4ab = (a+b)^2 - (a-b)^2$

$$\Rightarrow 4 \sum (\Delta_i X)(\Delta_i Y) = \cancel{\sum (\Delta_i (X+Y))} = \sum (\Delta_i (X+Y))^2 - (\Delta_i (X-Y))^2.$$

$$\Rightarrow 4 \underbrace{[X, Y]}_{\text{Joint QV of } X \,\&\, Y} = \underbrace{[X+Y, X+Y]}_{\text{QV of } X+Y} - \underbrace{[X-Y, X-Y]}_{\text{QV of } X-Y.}.$$

Proof: (Product Rule). X, Y — 2 processes.

$$d(XY) = X\,dY + Y\,dX + d[X,Y].$$

Pf:

$$d(X+Y)^2 \overset{\text{Itô}}{=} 2(X+Y)\,d(X+Y) + \frac{1}{2} \cdot 2\, d[X+Y, X+Y].$$

$$d(X-Y)^2 \overset{\text{Itô}}{=} 2(X-Y)\,d(X-Y) + d[X-Y, X-Y].$$

Subtract:

$$d(4XY) = 4X\,dY + 4Y\,dX + 4\,d[X,Y].$$

$$\Rightarrow d(XY) = \cancel{4}X\,dY + Y\,dX + d[X,Y]$$

Rules to compute Joint QV.

① Say $B \longrightarrow$ cts & finite first variation. (adapted).

$X \longrightarrow$ cts, adapted.

Then $[X, B] = 0 = [B, X]$.

<u>Check:</u>

$$4[X,B] = \underbrace{[X+B, X+B]}_{[X,X]} - \underbrace{[X-B, X-B]}_{[X,X]} = 0. \quad //$$

Thm: Multi Dim Itô formula (Itô Doeblin formula).

$X_1, X_2, \cdots X_n \longrightarrow n$ – stochastic processes.

Let $X = (X_1, X_2, \cdots X_n)$. [random vector].

$f = f(t, x_1, x_2, \cdots x_n)$ is a fn. $\left(\begin{array}{l} x_1, \cdots x_n \in \mathbb{R}. \\ \text{Let } x = (x_1, \cdots x_n) \end{array} \right)$.

1. $\partial_t f$ exists and is cts.
2. $\partial_i \partial_j f$ all exist & are cts $(i, \& j \in \{1, \cdots n\})$.

$\hookrightarrow \dfrac{\partial^2 f}{\partial x_i \partial x_j}$.

$$\partial_i f = \frac{\partial f}{\partial x_i}$$

Then $d\Big(f(t, X_1, X_2, \cdots X_n)\Big) = d\Big(f(t, X)\Big)$

$$= \partial_t f(t, X(t))\, dt + \sum_{i=1}^{n} \partial_i f(t, X)\, dX_i(t)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f(t, X(t))\, \underbrace{d[X_i, X_j](t)}_{\text{Joint QV.}}$$

$\left[\partial_i \partial_j f = \dfrac{\partial^2 f}{\partial x_i \partial x_j} = \cancel{f_{x_i x_j}}\right].$

Intuition: Taylor expand & follow the 1D argument

~~Prop~~: Recall: M is a cts mg $\Rightarrow M^2 - [M,M]$ is also a cts mg.

Joint QV: (1) Let M & N be two cts mg's.

$$E M(t)^2 < \infty \text{ \& } E N(t)^2 < \infty.$$

then $MN - [M,N]$ is again a cts mg.

(2) If B is any cts adapted process of finite 1st variation such that $MN - B$ is a mg & $B(0) = 0$,

then $B = [M,N]$.

Prop: X, Y, Z 3 processes. $\alpha \in R$.

Cont Q: Is QV of $(X+Y)$ = QV of X + QV of Y?

Yes.

No

$$[X, Y+\alpha Z] = [X,Y] + \alpha [X, Z]. \quad (\text{dist law}).$$

Prop: X_1, X_2 two Ito processes. $\sigma_1, \sigma_2 \rightarrow$ 2 adapted proc.

Let $I_1(t) = \int_0^t \sigma_1(s)\, dX_1(s)$ & $I_2(t) = \int_0^t \sigma_2(s)\, dX_2(s)$.

Then $[I_1, I_2](t) = \int_0^t \sigma_1(s)\sigma_2(s)\, d[X_1, X_2](s)$.

I.e. if $dI_1 = \sigma_1\, dX_1$ & $dI_2 = \sigma_2\, dX_2$.

then $d[I_1, I_2] = \sigma_1 \sigma_2\, d[X_1, X_2]$ . . . . $(*)$

Note: Special case. $X_1 = X_2$ & $\sigma_1 = \sigma_2$. $\Rightarrow I_1 = I_2$.

Know $d[I_1, I_1] = \sigma_1^2\, d[X_1, X_1]$. from before

$= \sigma_1 \sigma_1\, d[X_1, X_1]$ from $(*)$.

Intuition behind (*):

$$[I_1, I_2](T) = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (\Delta_i I_1)(\Delta_i I_2)$$

$$= \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \sigma_1 \, dX_1 \right) \left(\int_{t_i}^{t_{i+1}} \sigma_2 \, dX_2 \right).$$

$$= \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \sigma_1(t_i) \left(X_1(t_{i+1}) - X_1(t_i) \right) \sigma_2(t_i) \left(X_2(t_{i+1}) - X_2(t_i) \right)$$

$$= \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \sigma_1(t_i)\, \sigma_2(t_i)\, \underbrace{(\Delta_i X_1)(\Delta_i X_2)}_{\Delta_i [X_1, X_2].}.$$

$$= \int_0^T \sigma_1(t)\, \sigma_2(t)\, d[X_1, X_2](t).$$

__Prop:__ Say M & N are 2 cts mg's. $E\,M(t)^2 < \infty$ & $E\,N(t)^2 < \infty$.

If M & N are independent then $[M, N] = 0$.

__Warning:__ If M & N are 2 mg's (not ~~nec~~ cts) $E\,M(t)^2 < \infty$, $E\,N(t)^2 < \infty$.

& M & N are ind

$$\not\Rightarrow [M, N] = 0.$$

Try #1 at Proof: M & N are ind. & are mg's.

Show MN is a mg. $\Rightarrow [M, N] = 0$.

Know $E\,M(t)N(t) = E\,M(t)\;E\,N(t)$ (ind).

To check MN is a mg,

WRONG.

$$\boxed{E\left(M(t)N(t) \,\middle|\, \mathcal{F}_s\right) \overset{\text{indep}}{=\!=} E\left(M(t) \,\middle|\, \mathcal{F}_s\right) E\left(N(t) \,\middle|\, \mathcal{F}_s\right)}$$

$$= M(s)N(s),$$

$\Rightarrow$ MN is a mg $\Rightarrow$ $[M, N] = 0$.

M & N indep $\not\Rightarrow$ $E\left(M(t)N(t) \,\middle|\, \mathcal{F}_s\right) = E\left(M(t) \,\middle|\, \mathcal{F}_s\right) \cdot E\left(N(t) \,\middle|\, \mathcal{F}_s\right).$

Correct Proof: Assume M, N, cts mg's. & M, N indep.

NTS $[M, N] = 0$.

Will show $E\left([M,N](T)\right)^2 = 0 \Rightarrow$ $\Rightarrow$ Done.

$P = \{0 = t_0 < t_1 \dots < t_n = T\}$.

$$E\left(\sum_{i=0}^{n-1} (\Delta_i M)(\Delta_i N)\right)^2 = E \sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 \quad \text{.....} \; (*)$$

$$+ 2E \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} (\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N)$$

Claim: Note $E(\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N) \overset{\text{indep}}{=} \left[E(\Delta_i M)(\Delta_j M)\right]\left[E(\Delta_i N)(\Delta_j N)\right]$

$$= \left(E\, \Delta_i M\, \Delta_j M \right) E\, E \left((N(t_{i+1}) - N(t_i)) (N(t_{j+1}) - N(t_j)) \,\middle|\, \mathcal{F}_{t_j} \right)$$

$$= \left(E\, \Delta_i M\, \Delta_j M \right) E \left[(N(t_{i+1}) - N(t_i))\, \underbrace{E \left(N(t_{j+1}) - N(t_j) \,\middle|\, \mathcal{F}_{t_j} \right)}_{0 \ (\because N \text{ is a mg})} \right].$$

1^{st} term in (*) :

$$E \sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 \overset{\text{indep}}{=} \sum_{i=0}^{n-1} E(\Delta_i M)^2\, E(\Delta_i N)^2$$

$$\leq \underbrace{\left(\max_i E(\Delta_i M)^2 \right)}_{\substack{\downarrow \|P\| \to 0 \\ 0 \ (\because M \text{ is cts}).}} \underbrace{\sum_{i=0}^{n-1} E(\Delta_i N)^2}_{\substack{\downarrow \|P\| \to 0 \\ E\,[N,N](T)}}.$$

$\therefore$ From (*) : $$\underbrace{\lim_{\|P\| \to 0} E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2}_{E\left([M,N](T)\right)^2} = 0$$

$$\Rightarrow [M,N](T) = 0 \qquad //.$$

Multi Dim B.M.

Def: We say $W = (W_1, W_2, \dots W_d)$ is a std. d-dimen B.M. if ① Each W_i is a std 1 dim BM.
& ② If $i \neq j$, W_i & W_j are indep.

Note: $d[W_i, W_j](t) = \begin{cases} dt & i = j \\ 0\,dt & \text{if } i \neq j \end{cases}$

Theorem (Levy): Let $M = (M_1, M_2, \ldots M_d)$ be a cts d-dimensional mg.

~~Then~~ If $M(0) = 0$ &

$$d[M_i, M_j](t) = \begin{cases} dt & i = j \\ 0\,dt & i \neq j. \end{cases}$$

Then M is a std B.M.

[Reviews question 7.5],