Black-Scholes Extension

↳ pricing European options that pay dividends

i) Black-Scholes that you covered in class assumes that the stock price S_t does not pay dividends.

↳ We want to allow dividend payments because many real world stocks pay dividends.

Q: Do dividends matter for option pricing?

A: Yes! We price by replicating so we buy or sell the asset to recreate the option payoff. When I buy the stock I receive dividends, <u>but</u> the option is written on the stock value (i.e. it ignores dividends)

How do we model dividends?

First let's recall the assumptions of Black-Scholes.

1) S_t follows Geometric Brownian Motion

2) We have access to a Money-Market account that pays out at continuous risk-free rate r.

$$\left(dB_t = rB\,dt \right)$$

3) Can buy/sell fractions of stock.

4) Frictionless Market (i.e. no transaction costs).

5) No arbitrage in the market.

Now dividends:

6) We model dividends continuously at rate q.

(i.e. the dividend is proportionate to the current stock value).

If I observe market price of the stock then

(α)

(observed value) $dS_t = S_t(\mu - q)\,dt + \sigma S_t\,dW_t$

If I hold the stock I receive $d\tilde{S} = \mu S\,dt + \sigma S\,dW_t$

Total Return = Asset price fluctuations + Dividend payments.

i.e. $\tilde{S} = S + D$ where $dD_t = q S_t \, dt$

$\hookrightarrow$ portfolio price. $\hookrightarrow$ observed.

Important point: S_t is the traded one and is the one I will write the option on.

~~Reality~~ (Remarks:) 1) In reality dividends are paid out discretely. (typically quarterly).

2) We don't know future dividend schedules exactly. They have to be estimated.

3) If we reinterpret S as an index (say) then this assumption is more realistic.

Option Pricing:

Let's Derive a PDE for a European option where we denote its value by $V(x, t)$ that expries at time T.

x → current value. t → time (typically plug in $V(S_t, t)$).

Replication: Portfolio X_t → calculate it in 2 ways and compare

1. Buy Δ_t shares of stock S_t and I put the rest in the M-M account $(X_t - \Delta_t S_t)$.

$$dX_t = \Delta_t d\tilde{S}_t + r(X_t - \Delta_t S_t)dt$$

$$= \Delta_t \Big(\underbrace{S_t(\mu - q)dt + S_t \sigma dW_t}_{dS_t} + \underbrace{q S_t dt}_{dD_t} \Big)$$

$$+ r(X_t - \Delta_t S_t)dt$$

$$= \sigma \Delta_t S_t \, dW_t + \left(\Delta_t S_t \mu + r X_t - r \Delta_t S_t \right) dt$$

(2) We want $X_T = V(S_T, T)$. $(X_t = V(S_t, t))$

By Itô:

$$dV(S_t, t) = V_t \, dt + V_x \, dS_t + \frac{1}{2} V_{xx} \, d[S,S]_t$$

$$= \left(V_t + S_t(\mu - q) V_x + \frac{1}{2}\sigma^2 s^2 V_{xx} \right) dt$$

$$+ \sigma S_t V_x \, dW_t .$$

Compare (1) and (2) (i.e. dW_t and dt terms).

choose $\Delta_t = V_x$. (dW_t terms).

$$\& \quad V_x S_t \mu + rV - r \overset{V_x}{\cancel{\Delta_t}} S_t = V_t + S_t(\mu - q) V_x + \frac{1}{2}\sigma^2 s_t^2 V_{xx}$$

$$0 = V_t + (r-q) S_t V_x + \frac{1}{2}\sigma^2 S_t^2 V_{xx} - rV.$$

We want to find $v(x,t)$ which has to satisfy

$$\boxed{0 = V_t(x,t) + (r-q)\, x\, V_x(x,t) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x,t) - r\,v(x,t)}$$

$\forall$ $t \in [0,T)$ and $x \in [0,\infty)$

This is a <u>global rule</u>. This has to hold for all European options.

We append this PDE with a terminal condition.

$$\boxed{v(x,T) = P(x)} \rightsquigarrow \text{payoff of option.}$$

↳ Contract specific.

__Ex__ for call $p(x) = (x-K)^+$ is call

we also need Boundary conditions or growth conditions to solve this for a particular pay off

we have conditions at $v(0,t)$ and $v(\infty,t)$.

__Ex__ for call we have $v(0,t) = 0$.

$\cancel{v(\infty,t) = \lim_{x\to\infty} v(x,t) = (x-K) = (x-K)^+}$

$$\lim_{x\to\infty} v(x,t) - (x-K) = 0.$$

$$\Updownarrow$$

$$\Big(\underbrace{v_x}_{\text{Delta}}(x,t) = 1 \Big).$$

Let's build our intuition about dividends:

Ex Consider a prepaid forward on S_t.

i.e. I pay you now some price

and at time T I for sure receive S_T.

i.e. one share of the stock.

Case 1 S pays no dividends.

price is just S_t. $\rightsquigarrow$ I replicate by buying 1 share now.

(check B-S formula with $r = 0$) $\Delta = 1$.

Case 2: S pays dividends.

price is $S_t e^{-q(T-t)}$; $\rightarrow$ reinvested will accumulate to $e^{q(T-t)} S_t$.

$$\Delta_t = e^{-q(T-t)}$$ $\rightsquigarrow$ tailed position

Back to our setting:

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + (r-q)\cancel{x}V_x - rV = 0. \quad (*)$$

$V \to$ option price for dividend paying stock

Let $u \to$ option price (same option) but written on a non-dividend paying stock.

i.e. u satisfies regular Black-Scholes PDE

claim Then $V(x,t) = u(x e^{-q(T-t)}, t)$ will satisfy $(*)$.

pf: $V_t(x,t) = u_x(xe^{-q(T-t)}, t)\, x q\, e^{-q(T-t)} + u_t(xe^{-q(T-t)}, t).$

$$V_x(x,t) = u_x(xe^{-q(T-t)}, t)\, e^{-q(T-t)}$$

$$V_{xx}(x,t) = u_{xx}(xe^{-q(T-t)}, t)\, e^{-2q(T-t)}$$

set $\bar{x} = x e^{-q(T-t)}$

LHS of 6 :

$$= u_x \cancel{(\bar{x}, t)\, q}\, e^{-q(T-t)} x + u_t(\bar{x}, t) + \frac{1}{2}\sigma^2 x^2 e^{-2q(T-t)} u_{xx}(\bar{x}, t)$$

$$+ (r - \cancel{q})\, \cancel{x}\, e^{-q(T-t)} u_x(\bar{x}, t) - r u(\bar{x}, t)$$

$$= u_t(\bar{x}, t) + \frac{1}{2}\sigma^2 \Big(\underbrace{x e^{-q(T-t)}}_{\bar{x}}\Big)^2 u_{xx}(\bar{x}, t) + r\Big(\underbrace{x e^{-q(T-t)}}_{\bar{x}}\Big) u_x(\bar{x}, t) - r u(\bar{x}, t)$$

$$= u_t(\bar{x}, t) + \frac{1}{2}\sigma^2 \bar{x}^2 u_{xx}(\bar{x}, t) + r \bar{x} u_x(\bar{x}, t) - r u(\bar{x}, t)$$

$$= 0 = \text{RHS}. \quad \square$$

Ex Call option. here

$$u(x,t) = x\,N(d_1) - K e^{-r(T-t)} N(d_2).$$

call price on non-dividend paying stock

$$d_1 = \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \cancel{d_1 - \sigma\sqrt{T-t}}\ d_1 - \sigma\sqrt{T-t}$$

$$V(x,t) \underset{q}{=} u\left(x e^{-q(T-t)}, t\right)$$

call price on dividend paying stock

$$= x e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2).$$

$$d_1 = \frac{\ln\left(\frac{x e^{-q(T-t)}}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\ln\left(\frac{x}{K}\left(e^{-q(T-t)}\right)\right) = \ln\left(\frac{x}{K}\right) + \ln\left(e^{-q(T-t)}\right)$$

$$= \ln\left(\frac{x}{K}\right) - q(T-t).$$