Last time: B-S-M formula.

$S(t)$. $\boxed{dS = \alpha S\,dt + \sigma S\,dW}$

Market $\begin{cases} \to \text{Stock} \to \text{price modelled by GBM}(\alpha, \sigma) \\ \to \text{Money Market} \longrightarrow \text{intrest rate } r. \end{cases}$

European call strike K mat T.

<u>Theorem:</u> (1) If $c(t, S(t))$ is the AFP of the call then

(a) $\partial_t c + r x \partial_x c + \frac{\sigma^2}{2} x^2 \partial_x^2 c = rc \leftarrow$ PDE

(b) $c(t, 0) = 0$

(c) $c(T, x) = (x - K)^+$,

$\left.\right\} \leftarrow$ Boundary conditions. (terminal condition).

Thm ② : Conversely if c satisfies ⓐ $\longrightarrow$ ⓒ then

$$c(t, S(t)) = \text{AFP of the call option.}$$

Can solve ⓐ $\longrightarrow$ ⓒ & get.

$$c(t,x) = x\,N(\underbrace{d_+}_{d_+(T-t,x)}) - K e^{-r(T-t)} N(\underbrace{d_-}_{d_-(T-t,x)})$$

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right).$$

$$N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}.$$

Proof of Thm 2:

Construct a portfallio: $X(t)$ = wealth of the Pf.

$\Delta(t)$ shares of stock

Rest cash $X(t) - S(t)\Delta(t)$

Start with $X(0) = c(0, S(0))$.

Choose $\Delta(t) = \partial_x c(t, S(t))$ ("delta Hedging rule").

Claim: $X(t) = c(t, S(t))$ for all $t \geq 0$

$\boxed{\text{Trick: Discount.}}$

Let $Y(t) = e^{-rt} X(t)$

Compute dY: Itô.

$$dY = -re^{-rt} X\,dt + e^{-rt} dX + 0$$

$$= -rY\,dt + e^{-rt}\left(\Delta(t)\,dS + r(X - S\Delta)\,dt \right)$$

$$= e^{-rt} \Delta(t)\,dS - re^{-rt} S\Delta\,dt \quad (*).$$

Also compute $d\left(e^{-rt} c(t, S(t))\right)$:

$$
\begin{aligned}
d\left(e^{-rt} c(t, S(t))\right) &= \left(-r e^{-rt} c + e^{-rt} \partial_t c\right) dt \\
&\quad + e^{-rt} \partial_x c \, dS + \frac{1}{2} e^{-rt} \partial_x^2 c \, \sigma^2 S^2 \, dt \\
&= e^{-rt} \underbrace{\left(\partial_t c + \frac{\sigma^2}{2} S^2 \partial_x^2 c - r c\right)} dt + e^{-rt} \partial_x c \, dS \\
&= e^{-rt} \left(-r S \underbrace{\partial_x c}_{\Delta}\right) dt + e^{-rt} \underbrace{\partial_x c}_{\Delta} \, dS
\end{aligned}
$$

(delta Hedging),

$$
= dY \quad \text{by } (*)
$$

$$\Rightarrow d\left(e^{-rt}c(t,S(t)) - e^{-rt}X(t)\right) = 0$$

$$\Rightarrow e^{-rt}c(t,S(t)) - e^{-rt}X(t) - \underbrace{\left(c(0,S(0)) - X(0)\right)}_{0 \text{ by assumptions}} = 0$$

$$\Rightarrow e^{-rt}\left(c(t,S(t)) - X(t)\right) = 0$$

$$\Rightarrow c(t,S(t)) = X(t). \qquad \text{for all } t < T$$

$$\left.\begin{array}{l} \Rightarrow \underbrace{c(T,S(T))}_{(S(T)-K)^+} = X(T) \end{array}\right\} \Rightarrow X \text{ is a Replicating Pf.}$$

$$\Rightarrow X(t) = AFP$$

$$\Rightarrow c(t,S(t)) = AFP \quad //.$$

Remark: What if we try & prove Thm ② without discounting.

Choose $X(0) = c(0, S(0))$ & $\Delta(t) = \partial_x c(t, S(t))$.

$$dX = \Delta \, dS + r(X - \Delta S) \, dt$$

$$d\, c(t, S(t)) = \partial_t c \, dt + \partial_x c \, dS + \frac{1}{2} \partial_x^2 c \, \sigma^2 S^2 \, dt$$

$$= \left(\partial_t c + \frac{1}{2} \sigma^2 S^2 \partial_x^2 c \right) dt + \partial_x c \, dS$$

$$\overset{ⓐ}{=} (rc - rS \partial_x c) \, dt$$

$$\Rightarrow d\left(X - c(t, S(t))\right) = r(X - c) \, dt$$

$$\Rightarrow \frac{d}{dt}\left(X - c\right) = r\,(X - c)$$

$$\Rightarrow X(t) - c(t, S(t)) = \underbrace{\Big(X(0) - c(0, S(0))\Big)}_{0} e^{rt}.$$

$$\Rightarrow X(t) = c(t, S(t))$$ & finish proof.

Q: AFP price of a put?

Payoff at maturity $= (K - x)^+$

Put call parity.

Buy 1 call short 1 put.

$$c(t, S(t)) - p(t, S(t)) = X(t).$$

At maturity $X(T) = (S(T)-K)^+ - (K-S(T))^+$

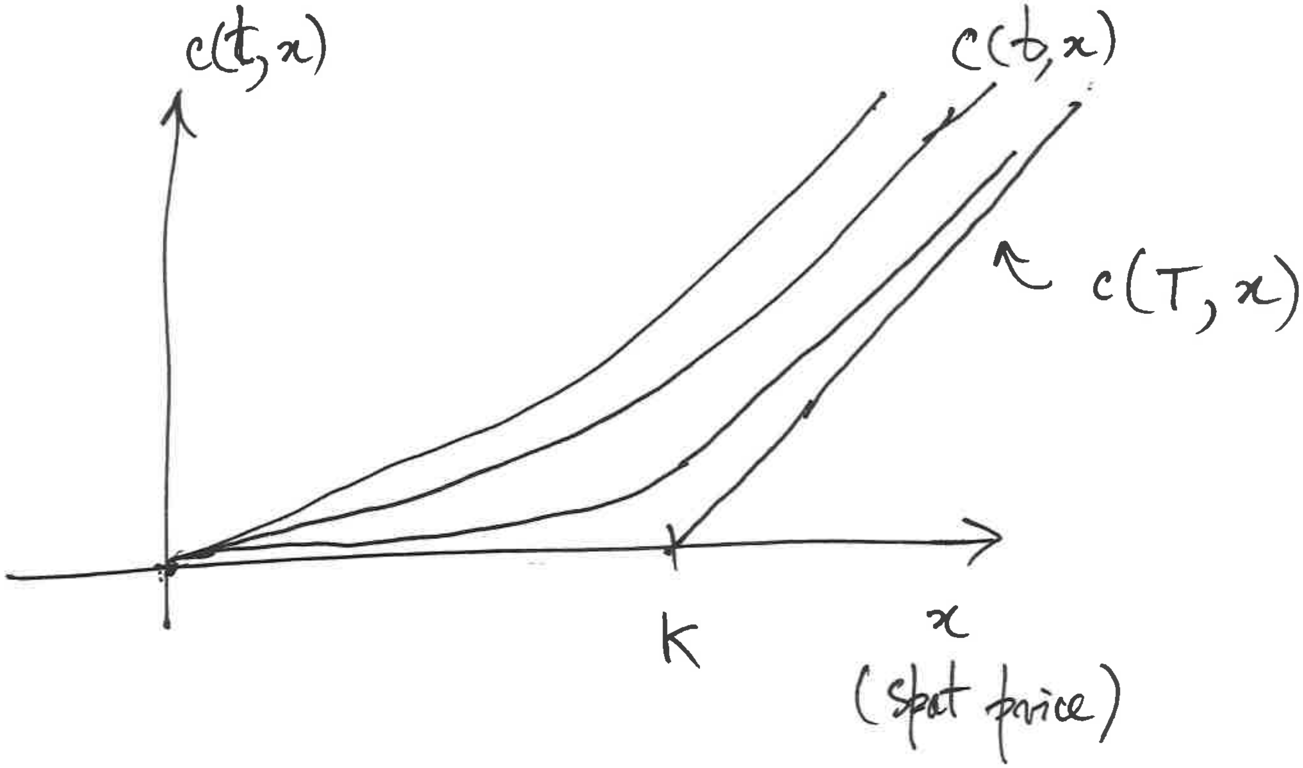
$$= S(T) - K. \quad \text{(Forward Contract)}.$$

Hedge a forward contract by buying 1 share of stock & borrowing $\cancel{e^{-rT}K}$ $\cancel{e^{-rt}K}$ $e^{-r(T-t)}K$ at time t.

$$\Rightarrow c(t, S(t)) - p(t, S(t)) = S(t) - e^{-r(T-t)}K$$

$$\Rightarrow p(t, S(t)) = c(t, S(t)) - S(t) + e^{-r(T-t)}K.$$

Properties of $c(t, x) = x\,N(d_+) - K e^{-r(T-t)} N(d_-)$



$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(\cancel{\frac{\sigma^2}{2} \pm}\, r \pm \frac{\sigma^2}{2}\right)\tau\right).$$

Greeks: Derivatives of c wrt t, & x.

① Delta: $\partial_x c$ is called the Delta.

(Delta Hedging: R-Pf holds exactly $\partial_x c(t, S(t))$ shares of S at time t.)

Compute $$\partial_x c = \partial_x \left(x N(d_+) - K e^{-r(T-t)} N(d_-) \right).$$

$$= N(d_+) + \underbrace{x N' d_+' - K e^{-r\tau} N'(d_-) d_-'}_{\text{cancel (You compute \& check).}}$$

$$= N(d_+)$$

② <u>Gamma:</u> $\partial_x^2 c$ is called the Gamma.

$$\partial_x^2 c = N'(d_+) \cdot d_+' = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{d_+^2}{2}\right).$$

③ <u>Theta:</u> $\partial_t c$ is called the Theta.

$$\partial_t c = -rKe^{-r\tau} N(d_-) - \frac{\sigma x}{2\sqrt{\tau}} N'(d_+)$$

<u>Prop:</u> ① c is increasing as a fn of x ($\because \partial_x c > 0$)

② c is decreasing as a fn of t ($\because \partial_t c < 0$).

③ c is convex as a fn of x ($\because \partial_x^2 c > 0$).

Hedging a Short Call: Sell 1 call

Get $c(t,x)$ cash. $\longrightarrow$ Hedge the call with this cash.

Buy $\Delta(t) = \partial_x c(t,x)$ shares of stock (spot price x).

Rest Cash $= c(t,x) - x\,\partial_x c(t,x)$

$$= \cancel{x N(d_+)} - K e^{-r\tau} N(d_-) - \cancel{x N(d_+)}$$

$$= -K e^{-r\tau} N(d_-) < 0$$

Δ Neutral / Long Gamma:

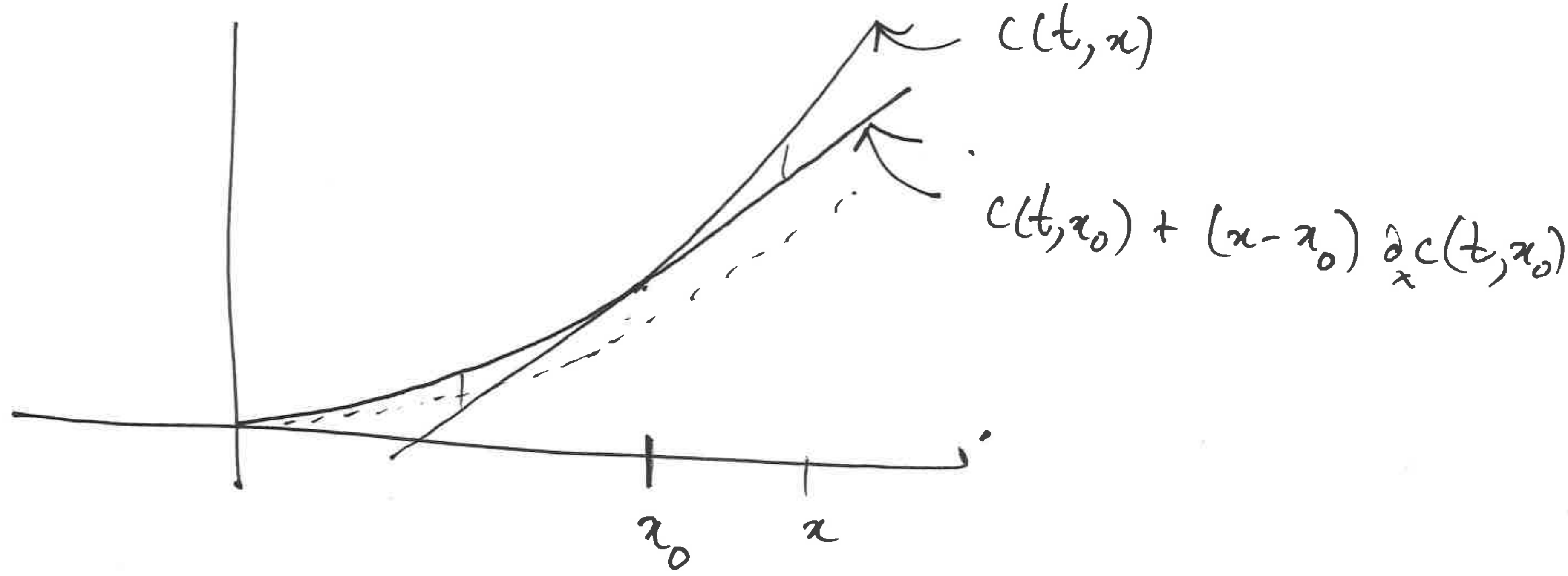
Say at time t price $= x_0$

Short $\partial_x c(t, x_0)$ shares & buy 1 call valued at $c(t, x_0)$.

Balance $= M = x_0 \partial_x c(t, x_0) - c(t, x_0)$.

Say spot price changes to x (Hold position on stock).

$$
\begin{aligned}
\text{Pf value} &= c(t,x) - \partial_x c(t, x_0)\, x + M \\
&= c(t,x) - \partial_x c(t, x_0)\, x + x_0 \partial_x c(t, x_0) - c(t, x_0) \\
&= c(t,x) - \Big[\underbrace{c(t, x_0) + (x - x_0)\, \partial_x c(t, x_0)}_{\text{tangent line } (\Delta \text{ neutral})}\Big]
\end{aligned}
$$



Long Gamma: the above Pf seems +ve balance even if $x > x_0$ or $x < x_0$.