Two processes:

$X(t)$: replicating portfolio process

$C(t, S(t))$: call price process.

$$dX(t) = \Delta(t) \cancel{ds}\, dS(t) + \big(X(t) - \Delta(t)S(t)\big)\, r\, dt.$$

$dC(t, S(t))$: use Ito's lemma

match dt term and $dW(t)$ term of these two differentials. By matching $dW(t)$ term, you get ~~Delta~~ Delta-hedging rule. By matching dt term, get PDE.

If a call is priced with volatility σ_1, and the realized vol is σ_2. Try to find an arbitrage opportunity.

if $\sigma_1 < \sigma_2$.

when the call is priced: $dS(t) = S(t)\alpha dt + \sigma_1 S(t) \cancel{dt} dW(t)$

The actual stock price follows: $dS(t) = S(t)\alpha dt + \sigma_2 S(t) \cancel{dt} dW(t)$

Idea: Long the call since the call is priced cheaper,

short the replicating portfolio.

The replicating portfolio of a call:

$(X(t))$

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$$dX(t) = \Delta(t)\, \underset{dS(t)}{\cancel{S(t)dt}} + r\,(X(t) - \Delta(t)S(t))\,dt.$$

$$= \underbrace{\Delta(t)}_{C_x}\left[\alpha S(t)dt + \sigma_2 S(t)dW(t)\right] + r\left(X(t) - \Delta(t)S(t)\right)dt.$$

$$= \left[C_x \alpha S(t) + rX(t) - rC_x S(t)\right]dt + C_x \sigma_2 S(t)\,dW(t)$$

To get rid of $rX(t)dt$ on r.h.s, consider discounting $X(t)$ with e^{-rt}, calculate $de^{-rt}X(t)$:

$$de^{-rt}X(t) = e^{-rt}(-rX(t)dt + dX(t))$$

$$= e^{-rt}\left[(C_x \alpha S(t) - r C_x S(t))\,dt + C_x \sigma_2 S(t)\,dW(t)\right]$$

Next, consider $dC(t, S(t))$:

$$dC(t,S(t)) = C_t\,dt + C_x\,dS(t) + \frac{1}{2} C_{xx} \underbrace{(dS(t))^2}_{d[S,S](t)}$$

$$= C_t\,dt + C_x\left[\alpha S(t)dt + \sigma_2 S(t)\,dW(t)\right]$$

$$+ \frac{1}{2} C_{xx} \sigma_2^2 S(t)^2\,dt$$

$$= \left[C_t + C_x \alpha S(t) + \frac{1}{2} C_{xx} \sigma_2^2 S(t)^2\right] dt + \sigma_2 C_x S(t)\,dW(t)$$

④

$$de^{-rt}c(t,S(t)) = e^{-rt}[-rc\,dt + dc(t,S(t))]$$

$$= e^{-rt}[(-rc + c_t + c_x\alpha S(t) + \tfrac{1}{2}c_{xx}\sigma_2^2 S(t)^2)\,dt$$

$$+ \sigma_2 c_x S(t)\,dW(t)]$$

$$d(e^{-rt}c(t,S(t)) - e^{-rt}X(t)) = e^{-rt}[(-rc + c_t + \cancel{c_x\alpha S(t)} + \tfrac{1}{2}c_{xx}\sigma_2^2 S(t)^2)$$

$$dt + \cancel{\sigma_2 c_x S(t)\,dW(t)} - (\cancel{c_x\alpha S(t)} - rc_x S(t))\,dt - \cancel{c_x\sigma_2 S(t)\,dW(t)}]$$

$$= e^{-rt}[-rc + c_t + \tfrac{1}{2}c_{xx}\sigma_2^2 S(t)^2 + rS(t)c_x]\,dt.$$

Black-Scholes PDE:

$$c_t + rxc_x + \tfrac{1}{2}c_{xx}x^2\sigma_1^2 = rc.$$

$$= e^{-rt}\Big[\cancel{-C_t} - \cancel{rS(t)C_x} - \frac{1}{2} C_{xx} S(t)^2 \sigma_1^2$$

$$+ \cancel{C_t} + \frac{1}{2} C_{xx} \sigma_2^2 S(t)^2 + \cancel{rS(t)C_x} \Big] dt.$$

$$= \cancel{e^{-rt}[}\ \underbrace{\frac{e^{-rt}}{2} C_{xx} S(t)^2 (\sigma_2^2 - \sigma_1^2)}_{>0}\ dt.$$

because $\sigma_2^2 > \sigma_1^2$ and $C_{xx} > 0$.