Black Sholes Merton

① Model Stock price by Geometric B. M.

GBM → HW.

$S(t)$ = price of stock at time t.

$$dS(t) = \alpha S\, dt + \sigma S\, dW(t)$$

mean return rate. % volatility

$\alpha, \sigma \in \mathbb{R}$ (constants, parameters).

$$\Rightarrow S(t) = S(0) + \int_0^t \alpha S(s)\, ds + \int_0^t \sigma S(s)\, dW(s).$$

Let $Y(t) = \ln(S(t)) = f(t, S(t))$

Apply Itô. where $f(t, x) = \ln x$.

$$\partial_t f = 0, \quad \partial_x f = \frac{1}{x}, \quad \partial_x^2 f = -\frac{1}{x^2}.$$

$$dY = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2} \partial_x^2 f \, d[S, S].$$

$$= 0 + \frac{1}{S(t)} \left(\alpha S \, dt + \sigma S \, dW \right) - \frac{1}{2S^2} \sigma^2 S^2 \, dt$$

$$= \left(\alpha - \frac{\sigma^2}{2} \right) dt + \sigma \, dW$$

$$\Rightarrow \ln S(t) - \ln S(0) = \int_0^t \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t).$$

$$\Rightarrow S(t) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right).$$

GBM $\rightarrow$ $\alpha = \frac{\sigma^2}{2}$ then $\ln S(t)$ is a B. M.

Notation GBM(α, σ) is the process S.

$$\boxed{\int_0^t \sigma \, dW(s) = \sigma \int_0^t dW(s) = \sigma\left(W(t) - W(0)\right) = \sigma W(t)}$$

Setup:

Market → ① M.M. account   interest rate r.
Market → ② Stock → price is modelled by $GBM(\alpha, \sigma)$

③ European call · strike K, maturity T.

Q: What is the arbitrage free price of this option.
(B-S.).

Obs 1: AFP is a fn of t & spot price $S(t)$.
(& model parameters $\boxed{\alpha}$, r, σ).
T, K.

Theorem (B.S.M.). AF market.

Money market acct with interest rate r.

Stock $\to$ GBM(α, σ).

Eur call strike K mat T.

① If $c = c(t, x)$ is a fn such that when $t \leq T$, AFP of the call is $c(t, S(t))$.

then ⓐ $\partial_t c + r x \partial_x c + \frac{\sigma^2}{2} x^2 \partial_x^2 c = r c \leftarrow$

ⓑ $c(t, 0) = 0$

ⓒ $c(T, x) = (x - K)^+$

Thm part 2

Conversely if c is a function which satisfies (a), (b) & (c) then $c(t, S(t))$ is the arbitrage free price of the call!

Assumptions: 1 Frictionless. (no transaction cost).

2 Liquidity (buy & sell fractions of the asset).

3 Borrow & lend at rate r (constant).

Remark: Eq (a) (b) & (c) are called the Black-Scholes-Merton PDE.

(partial diff eq.).

Can solve B-S PDE. Sol is given by

$$c(t,x) = x\,N(d_+(T-t,x)) - K e^{-r(T-t)} N(d_-(T-t,x)).$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy \qquad (\text{CDF of } N(0,1)).$$

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right),$$

Proof Strategy:

Construct a replicating Portfolio.

$X(t) \longrightarrow$ wealth at time t.

- $\Delta(t)$ shares in stock.
- Rest cash.

Will find a portfolio such that

$$X(T) = (S(T) - K)^+$$

then $X(t) =$ AFP of the call.

Note: ~~Will see $\Delta(t) = \partial_x c(t, S(t))$.~~

$$\boxed{\Delta(t) = \partial_x c(t, S(t))}$$ Delta Hedging Rule.

Use Ito & "Uniqueness of the semi-mg decomp".

If $$Y(t) = Y(0) + \int_0^t b_1(s)\,ds + \int_0^t \sigma_1(s)\,dW(s)$$

& $$= Y(0) + \int_0^t b_2(s)\,ds + \int_0^t \sigma_2(s)\,dW(s)$$

then $b_1 = b_2$ & $\sigma_1 = \sigma_2$.

Reason: Above means.

$$\int_0^t (b_1(s) - b_2(s))\, ds = \int_0^t (\sigma_2(s) - \sigma_1(s))\, dW(s).$$

$\Rightarrow$ RHS is a mg! $\Rightarrow$ LHS is a mg.

But QV of LHS $= 0$.

Let $B(t) = \int_0^t (b_1(s) - b_2(s))\, ds$.

B an mg & $[B, B] = 0 \Rightarrow B^2 - [B, B]$ is a mg

$\Rightarrow B^2$ is a mg $\Rightarrow$

$$\Rightarrow E\,B(T)^2 = \underbrace{E\left(B(0)^2\right)}_{0.} \quad \text{for every } T \geq 0.$$

$$\Rightarrow E\;B(T)^2 = 0 \quad \Rightarrow B(T)^{\cancel{2}} = 0$$

$$\Rightarrow b_1 = b_2. \quad \Rightarrow \sigma_1 = \sigma_2.$$

//.

Means for us:

$$\text{If} \quad \left.\begin{aligned} dX &= b_1\,dt + \sigma_1\,dW \\ &= b_2\,dt + \sigma_2\,dW \end{aligned}\right\} \Rightarrow \begin{array}{l} b_1 = b_2 \\ \& \ \sigma_1 = \sigma_2. \end{array}$$

Pf of Thm Part 1.

Assume $c(t, S(t))$ = AFP of the call.

NTS c satisfies (a) (b) & (c).

needs work.

$X(t)$ = value of Rep Pf at time t. $= c(t, S(t))$.

Rep Pf $\to$ $\Delta(t)$ shares of stock.

$\to$ $X(t) - \Delta(t) S(t)$ in cash.

Compute dX in 2 diff ways.
Equate dt & dW terms.

$dS = \alpha S\,dt + \sigma S\,dW$

① $dX = \Delta(t)\,dS(t) + r\left(X(t) - \Delta(t)S(t)\right)dt$

② $= \Delta(t)\,\alpha S\,dt + \Delta(t)\,\sigma S(t)\,dW(t)$

$+ r(X - \Delta S)\,dt$

$$\boxed{dX = \left[\alpha\Delta S + r(X - \Delta S)\right]dt + \sigma\Delta S\,dW(t)}$$

(*)

(2) $X(t) = c(t, S(t))$ & Itô.

$$dX = \partial_t c \, dt + \partial_x c \, dS + \frac{1}{2}\partial_x^2 c \, d[S,S].$$

$$= \partial_t c \, dt + \partial_x c \left(\alpha S \, dt + \sigma S \, dW \right)$$
$$+ \frac{1}{2}\partial_x^2 c \, \sigma^2 S^2 \, dt$$

$$\boxed{dX = \left(\partial_t c + \partial_x c \, \alpha S + \frac{\sigma^2}{2}\partial_x^2 c \, S^2 \right) dt + \partial_x c \, \sigma S \, dW \quad (**)}$$

Equate ~~dt~~ dW terms in $\cancel{(**)}$ $(*)$ & $(**)$.

$$\Rightarrow \sigma \Delta(t) S(t) = \partial_x c(t, S(t)) \sigma S(t)$$

$$\boxed{\Rightarrow \Delta(t) = \partial_x c(t, S(t))}$$ ← Delta Hedging!

Equate dt terms:

$$\partial_t c + \partial_x c \, \alpha S + \frac{\sigma^2}{2} \partial_x^2 c \, S^2 = \alpha \Delta S + r(X - \Delta S).$$

Substitute
$$X = c$$
$$\Delta = \partial_x c$$
$$S = x$$

$$\Rightarrow \partial_t c + \cancel{\alpha x \partial_x c} + \frac{\sigma^2}{2} x^2 \partial_x^2 c = \cancel{\alpha \partial_x c\, x}$$

$$+ r\left(c - \partial_x c\, x\right)$$

$$\boxed{\Rightarrow \partial_t c + r x \partial_x c + \frac{\sigma^2}{2} x^2 \partial_x^2 c = r c}$$

eqn ⓐ

as promised!

Thm ~~a~~ part 2 Converse:

Assuming $c(t, x)$ satisfies the B.S. PDE.

NTS $c(t, S(t))$ is the AFP.

Let X = wealth of a portfolio → choose $\Delta(t) = \partial_x c$.

with → $\partial_x c(t, S(t))$ shares of Stock.

→ Rest cash.

Start with $X(0) = c(0, S(0))$.

<u>Claim:</u> X is the replicating portfolio & $X(t) = c(t, S(t))$. $\Rightarrow c(t, S(t)) = \text{AFP}$

$\Rightarrow$ Done.

Claim: first $X(t) = c(t, S(t))$ for all $t < T$

Trick: Let $Y(t) \equiv e^{-rt} X(t)$ (discounting).

$$dY = -r e^{-rt} X(t)\, dt + e^{-rt} dX$$

$$= -r\, Y(t)\, dt + e^{-rt}\left(\Delta(t)\, dS + r\left(X - \Delta S\right) dt\right).$$

$$= -r\, Y(t)\, dt + e^{-rt}\left(\partial_x c\, \alpha S\, dt + \partial_x c\, \sigma S\, dW\right).$$

$$\underbrace{+\, e^{-rt}\left(r\left(X - \partial_x c\, S\right)\right) dt}_{①}.$$

Ito: compute $d\left(e^{-rt}c(t,S(t))\right)$.

$$d\left(e^{-rt}c(t,S(t))\right) = \left(-r e^{-rt} c + \partial_t c\right) dt + e^{-rt}\partial_x c\, dS + \frac{1}{2} e^{-rt}\partial_x^2 c\, d[S,S].$$

$$= \left(-r e^{-rt} c + e^{-rt}\partial_t c\right) dt$$

$$+ e^{-rt}\partial_x c\, \cancel{S}\, \alpha S\, dt + e^{-rt}\partial_x c\, \sigma S\, dW$$

$$+ \frac{1}{2} e^{-rt}\partial_x^2 c\, \sigma^2 S^2\, dt$$

$$= -r\cancel{S}\,\underbrace{\partial_x c\, e^{-rt}\, dt + e^{-rt}\partial_x c\, \alpha S\, dt}_{2} + e^{-rt}\partial_x c\, \sigma\, dW$$

Claim: ① & ② are the same.

$$\Rightarrow dY = d\left(e^{-rt} c(t, S(t))\right).$$