(0) Last time:

If $dX = b\,dt + \sigma\,dW$ $\left(\begin{array}{l} (b, \sigma \text{ adapted proc.}) \\ E\int_0^t \sigma(s)^2 ds < \infty. \end{array} \right)$.

<u>Claim:</u> X is a mg

$\iff b = 0$ always.

Reason: Let $B(t) = \int_0^t b(s)\,ds$, $M(t) = \int_0^t \sigma(s)\,dW(s)$.

Say X is a mg. NTS: $B(t) = 0$ for every t.
($\iff b(t) = 0$ " " "),

Note M is a mg. $X = B + M \Rightarrow B = X - M$

$\Rightarrow B$ is a mg.

Also: $\left.[B,B](t) = 0\right\} \Rightarrow B^2 - \underbrace{[B,B]}_{0}$ is a mg.

$\Rightarrow$ B^2 is a mg.

$\Rightarrow$ $E\,B(t)^2 = E\,B(0)^2 = 0$

$\Rightarrow$ $B(t) = 0$ almost surely.

//.

Q1: $0 \le r < s < t$. Q: $E\big(W(r)W(s)W(t)\big) =$ ____

Guess: $\underline{\underline{0}}$

($\text{Know } E\,W(r)W(s) = r$).

Sol: $E\,W(r)W(s)W(t) = E\Big(W(r)W(s)\big(\underbrace{W(t)-W(s)}_{\text{ind}} + W(s)\big)\Big)$.

$= E\,W(r)W(s)^2 \;\ldots$ (solve).

Also $E\, W(r)W(s)W(t) = E\, E\big(W(r)W(s)W(t) \mid \mathcal{F}_s\big)$.

$$= E\, W(r)W(s)\, \underbrace{E\big(W(t) \mid \mathcal{F}_s\big)}_{W(s).}. = E\, W(r)\, W(s)^2$$

$$= E\,E\big(W(r)\,W(s)^2 \mid \mathcal{F}_r\big). = E\, W(r)\,\big\{E\big(W(s)^2 - s + s \mid \mathcal{F}_r\big)$$

$$= E\; W(r)\big(W(r)^2 - r + s\big) \quad (\because W(t)^2 - t \text{ is a } \underline{\text{mg}}),$$

$$= 0, \quad //$$

$\boxed{7.2}$ $X(t) = \int_0^{W(t)} e^{-s^2}\, ds.$ Q: find s-mg decomp.

$$X = X(0) + \underbrace{B}_{\text{bdd var (finite } 1^{st} \text{ var)}} + \underbrace{M}_{\text{mg}}.$$

Itô formula: $X(t) = f(t, W(t))$.

$$f(t,x) = \int_0^x e^{-s^2}\, ds.$$

$\partial_t f = 0$

$\partial_x f = e^{-x^2}$

$\partial_x^2 f = -2x\, e^{-x^2}.$

$$dX = \partial_t f\, dt + \partial_x f\, dW + \frac{1}{2}\partial_x^2 f\, dt$$

$$= 0 + e^{-W(t)^2} dW + \frac{1}{2}\left(-2W(t)\, e^{-W(t)^2}\right) dt,$$

$X(0) = 0$

$$B(t) = -\int_0^t W(s)\, e^{-W(s)^2} \underbrace{ds}_{R\text{- int}}$$

$$M(t) = \int_0^t e^{-W(s)^2} \underbrace{dW(s)}_{\text{Ito int.}}.$$

$$Y(t) = \exp\left(\int_0^t W(s)\, ds\right).$$: Q: Find the Itô decomp.

$B(t) =$ ↑ is already a process of finite 1st var.

(R. int).

Let $f(t, x) = \exp\left(\int_0^t W(s)\, ds\right)$.

Need 1 derivative in t } $\longrightarrow$ chain rule.

2 " in x. } ~~$W(t) = x$~~

$$\partial_t f = \exp(\quad) \cdot W(t).$$

$$\partial_x f = 0$$

$$\partial_x^2 f = 0.$$

Ito: $dY = \partial_t f \, dt + 0 \, dW + \frac{1}{2} 0 \, dt$

$$= W(t) \exp\left(\int_0^t W(s)\, ds\right) dt + 0$$

$$B(t) = \int_0^t \exp\left(\int_0^s W(r)\, dr\right) W(s)\, ds, \quad \& \quad M(t) = 0$$

$\boxed{7.3}$ $X(t) = \int_0^t W(s)\, ds \qquad Y(t) = \int_0^t W(s)\, \underline{\underline{dW(s)}}.$

Compute $\underline{E}\left(X(t) \mid \mathcal{F}_s\right)$ & $\underline{E}\left(Y(t) \mid \mathcal{F}_s\right)$.

① $\underline{\text{Claim:}}$ $E\left(Y(t) \mid \mathcal{F}_s\right) = Y(s)$ ($\because Y$ is a mg).

② $$E\left(\int_0^t W(r)\, dr \;\Big|\; \mathcal{F}_s\right) = E\left(\int_0^s (\;) \Big| \mathcal{F}_s\right) + E\left(\int_s^t W(r)\, dr \Big| \mathcal{F}_s\right).$$

IOU.

$$= \int_0^s W(r)\, dr + \int_s^t E\left(W(r) \mid \mathcal{F}_s\right) dr$$

$$= \int_0^s W(r)\, dr + \int_s^t W(s)\, dr = \int_0^s W(r)\, dr + (t-s) W(s).$$

Used: If b is some adapted process.

then $$E\Big(\underbrace{\int_0^t b(r)\,dr}_{\text{Riemann Int.}} \,\Big|\, \mathcal{F}_s\Big) = \int_0^t E\big(b(r) \,|\, \mathcal{F}_s\big)\,dr$$

$$\lim_{\|P\|\to 0} E\Big(\sum b(t_i)(t_{i+1}-t_i) \,\Big|\, \mathcal{F}_s\Big) = \lim_{\|P\|\to 0} \sum E\Big(b(t_i)(t_{i+1}-t_i)\,\Big|\,\mathcal{F}_s\Big)$$

$$= \lim_{\|P\|\to 0} \sum E\big(b(t_i) \,|\, \mathcal{F}_s\big)(t_{i+1}-t_i)$$

$$= \int_0^t E\big(b(r)\,|\,\mathcal{F}_s\big)\,dr$$

$\boxed{Q7.6}$ $X(t) = \int_0^t s \, dW(s)$. $\qquad Y(t) = \int_0^t W(s) \, ds$.

Q: Find $E X(t)^n$ & $E Y(t)^n$

<u>Claim:</u> $X(t)$ & $Y(t)$ are both normal!! (IOU).

Find moments using the MGF.

Q: Why is X normal?

$\boxed{\text{not random}}$.

$$X(t) \approx \underbrace{\lim_{\|P\| \to 0} \underbrace{\sum t_i \underbrace{(W_{t_{i+1}} - W_{t_i})}_{\text{normal}}}_{\text{normal.}}}_{\text{normal.}}$$

$$EX(t) = 0 \quad (\because X \text{ is a mg } \& \ X(0) = 0).$$

$$EX(t)^2 = E\left(\int_0^t s\, dW(s)\right)^2 \underset{\text{Itô isom}}{=} E\int_0^t s^2\, ds = \frac{t^3}{3}.$$

$$\Rightarrow X(t) \sim N(0, t^3/3).$$

$$\text{MGF of } X = \varphi_\lambda = E e^{\lambda X(t)} = \underline{e^{\frac{1}{2}\left(\frac{t^3}{3}\right)\lambda^2}}$$

$$EX(t)^n = \frac{d^n}{d\lambda^n}\varphi(\lambda)\Big|_{\lambda=0}$$

$$\text{Note } \varphi(\lambda) = e^{\frac{\lambda^2 t^3}{6}} = 1 + \frac{\lambda^2 t^3}{6} + \left(\frac{\lambda^2 t^3}{6}\right)^2 \cdot \frac{1}{2} + \left(\frac{\lambda^2 t^3}{6}\right)^3 \cdot \frac{1}{3!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\left.\frac{d^n \varphi}{d\lambda^n}\right|_{\lambda=0} = \begin{cases} 0 & n \text{ odd} \\ \frac{(2k)!}{k!}\left(\frac{t^3}{6}\right)^k & \cancel{n \text{ even}} \quad n = 2k. \end{cases} \leftarrow \text{ans.}$$

$$\varphi(\lambda) = \sum_{k=0}^{\infty} \left(\frac{\lambda^2 t^3}{6}\right)^k \cdot \frac{1}{k!}$$

$\boxed{Q\,7.6}$ $Y(t) = \int_0^t W(s)\,ds$. Find $E\,Y(t)^n$.

$W(s)$ ind $W(t)-W(s)$ & normal.

$$\begin{pmatrix} W(s) \\ W(t)-W(s) \end{pmatrix} \Rightarrow \begin{pmatrix} W(s) \\ W(t) \end{pmatrix} \text{ is } N$$

Claim $Y(t)$ is normal!

Reason:
$$\int_0^t W(s)\,ds = \lim_{\|P\|\to 0} \underbrace{\sum \underbrace{\underbrace{W(t_i)}_{\text{normal}}(t_{i+1}-t_i)}_{\text{normal.}}}_{\text{normal!}}$$

Compute $E\,Y(t)$ & $E\,Y(t)^2$.

① $E\,Y(t) = \int_0^t E\,W(s)\,ds = 0$

(2) $E\, Y(t)^2 = E\left(\int_0^t W(s)\,ds\right)^2$

Trick #1 : $E\left(\int_0^t W(s)\,ds\right)^2 = E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(s)\,ds\right).$

$$= E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(r)\,dr\right).$$

$$= E\int_{s=0}^{t}\int_{r=0}^{t} W(s)\,W(r)\;dr\cdot ds \qquad \left(EW(s)W(r) = s\wedge r\right)$$

$$= \int_{s=0}^{t}\int_{r=0}^{t} (s\wedge r)\;dr\,ds.$$ & evaluate this integral!..

$\boxed{Q7.5}$ Suppose $\sigma = \sigma(t)$ is $\boxed{\text{not random}}$

$$X(t) = \int_0^t \sigma(u)\, dW(u).$$

(a) $\lambda, s, t \in \mathbb{R}$, $(s<t)$. compute $E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$

Let $\varphi(t) =$

Apply Itô to $e^{\lambda X}$. $f(t,x) = e^{\lambda x}$.

$$\boxed{\begin{aligned} \partial_t f &= \cancel{\ldots}\ 0 \\ \partial_x f &= \lambda f \\ \partial_x^2 f &= \lambda^2 f. \end{aligned}}$$

$$d\left(e^{\lambda X(t)}\right) = 0\, dt + \partial_x f \cdot dX + \tfrac{1}{2}\partial_x^2 f\, d[X,X].$$

$$= \lambda e^{\lambda X(t)} \sigma(t)\, dW(t) + \tfrac{1}{2}\lambda^2 e^{\lambda X(t)} \sigma^2\, dt$$

Integrate from s to t.

$$e^{\lambda X(t)} - e^{\lambda X(s)} = \int_s^t (\ ) \, dW(r) + \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 e^{\lambda X(r)} \, dr.$$

$$\cancel{e^{\lambda(X(t)-X(s))}}$$

$$E(\quad | \mathcal{F}_s) = E(\quad | \mathcal{F}_s).$$

$$E\left(e^{\lambda X(t)} - e^{\lambda X(s)} \middle| \mathcal{F}_s\right) = \underset{(mg)}{0} + \frac{\lambda^2}{2} \int_s^t E\left(\sigma(r)^2 e^{\lambda X(r)} \middle| \mathcal{F}_s\right) dr.$$

$$\Rightarrow E\left(e^{\lambda(X(t)-X(s))} \middle| \mathcal{F}_s\right) - 1 = \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 E\left(e^{\lambda(X(r)-X(s))} \middle| \mathcal{F}_s\right) dr$$

$$\Rightarrow \varphi(t) - 1 = \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 \varphi(r) \, dr.$$

$$\Rightarrow \boxed{\varphi'(t) = \frac{\lambda^2}{2}\,\sigma(t)^2\varphi(t)} \Rightarrow \partial_t\varphi = \frac{\lambda^2}{2}\sigma(t)^2\varphi$$

$$\Rightarrow \frac{\partial_t\varphi}{\varphi} = \frac{\lambda^2}{2}\sigma(t)^2 \Rightarrow \partial_t\left(\ln\varphi\right) = \frac{\lambda^2}{2}\sigma(t)^2.$$

$$\Rightarrow \ln\varphi(t) - \ln\varphi(s) = \frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr.$$

$$\Rightarrow \varphi(t) = \underbrace{\varphi(s)}_{1}\cdot\exp\left(\frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr\right).$$

$$\Rightarrow \varphi(t) = E\left(e^{\lambda(X(t)-X(s))}\,\middle|\,\mathcal{F}_s\right) = \exp\left(\frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr\right).$$

(b) $\Rightarrow$ $X(t) - X(s)$ is ind of $\mathcal{F}_s$.

Gives Joint dist of $\left(X(r), X(t) - X(s)\right)$ to be normal.

& if $\sigma^2 = 1$ will get X to be BM..

Trick #2 to compute $E\left(\int_0^t W(s)\,ds\right)^2$.

Guess some fn $f(t, W(t))$ so that when you apply Itô, you get $W(s)\,ds$.

$$f(t, W(t)) = t\,W(t) \qquad \boxed{f(t,x) = tx. \quad \partial_t f = x, \quad \partial_x f = t, \quad \partial_x^2 f = 0}$$

$$d(tW(t)) = W(t)\,dt + t\,dW + 0.$$

$$tW(t) - 0 = \underbrace{\int_0^t W(s)\,ds}_{Y(t)} + \int_0^t s\,dW(s)$$

$$\int_0^t W(s)\,ds = tW(t) - \int_0^t s\,dW(s)$$ & use this to compute $E\,Y(t)^2$.

Q 7.7 $M(t) = \int_0^t W(s)\, dW(s)$

Q: Is M a mg? (Yes.)

Q: Is $M(t) - M(s)$ independent of $\mathcal{F}_s$? Guess

Yes 9+2

No 2+1(+1)

$$M(t) - M(s) = \int_s^t W(r)\, dW(r) \approx \lim_{\|P\| \to 0} \sum \underbrace{W(t_i)}_{\text{Not ind of } \mathcal{F}_s} \underbrace{(W(t_{i+1}) - W(t_i))}_{\text{ind of } \mathcal{F}_s}$$

Compute $E\left(\left(M(t)-M(s)\right)^2 \mid \mathcal{F}_s\right) \longleftarrow (*)$.

If $M(t)-M(s)$ was ind of $\mathcal{F}_s$, then

$$E\left(\left(M(t)-M(s)\right)^2 \mid \mathcal{F}_s\right) = E\left(M(t)-M(s)\right)^2 \longleftarrow (**)$$

Compute $(*)$ & hope $\neq (**)$

$(*)$: $$E\left(\left(M(t)-M(s)\right)^2 \mid \mathcal{F}_s\right) = E\left(M(t)^2 + M(s)^2 - 2M(t)M(s) \mid \mathcal{F}_s\right).$$

$$= E\left(M(t)^2 \mid \mathcal{F}_s\right) + M(s)^2 - 2M(s)\underbrace{E\left(M(t) \mid \mathcal{F}_s\right)}_{M(s)}$$

$$= E\left(M(t)^2 \mid \mathcal{F}_s\right) + M(s)^2 - 2M(s)^2$$

$$= E\left(M(t)^2 - M(s)^2 \mid \mathcal{F}_s\right).$$

Lets compute $E\left(M(t)^2 - M(s)^2 \mid \mathcal{F}_s\right)$:

$$d\, M(t)^2 = 2\, M(t)\, dM(t) + \frac{1}{2}\cdot 2\, d[M,M](t).$$

$$M(t)^2 - M(s)^2 = \int_s^t 2\, M(r)\, W(r)\, dW(r) + \int_s^t W(r)^2\, dr$$

$$\Rightarrow E\left(M(t)^2 - M(s)^2 \mid \mathcal{F}_s\right) = 0 + \int_s^t E\big(\underbrace{W(r)^2}_{-r+r} \mid \mathcal{F}_s\big)\, dr$$

$$= \int_s^t \left(W(s)^2 - s + r\right) dr = (t-s)W(s)^2 + \frac{(t-s)^2}{2}$$

$$\text{but } E\left(M(t)^2 - M(s)^2 \mid \mathcal{F}_s\right) = \frac{(t-s)^2}{2} + (t-s)\,W(s)^2 \leftarrow \text{random!}$$

$$\neq E\left(M(t)-M(s)\right)^2 \leftarrow \text{not random} \quad //$$

Q: Is $|W(t)| = \int_0^t \operatorname{sign}(W(s))\, dW(s)$?

Yes: 0

No: all.

~~Why?~~ Try Yes: Let $f(t,x) = |x|$. & apply Itô.

$\partial_t f = 0$ ✓

$\partial_x f = \operatorname{sign}(x) \qquad x \neq 0$

$\partial_x^2 f = 0 \qquad x \neq 0$.

Q: Can we Itô $f(t, W(t))$? Dont have 2 derivatives.

~~the~~ Only miss at $x=0$. Prob of ~~hitting~~ BM $= 0$ is 0

So maybe we can Itô any way?

If Yes: $d\,|W(t)| = \mathrm{sign}(W(t))\, dW(t) + 0$

$$\Rightarrow |W(t)| = \int_0^t \mathrm{sign}(W(s))\, dW(s).$$

(But this is FALSE!!

$$E\,|W(t)| = E\,|N(0,t)| = \sqrt{t}\, \underbrace{E\,|N(0,1)|}_{>0.}$$

But $E \int_0^t \mathrm{sign}(W(s))\, dW(s) = 0$

$$\Rightarrow |W(t)| \neq \int_0^t \mathrm{sign}(W(s))\, dW(s)$$

$$E|N(0,1)| = \int_{-\infty}^{\infty} \underline{|x|}\, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} > 0$$

$$\frac{d^n}{d\lambda^n}\left(\lambda^n\right)\Big|_{\lambda=0} = n!$$

$$(W(t)-W(0))\frac{d^3}{d\lambda^3}\lambda^3\Big|_{\lambda=0} = \cancel{\lambda} 3!$$

$$\underline{W(t)^3} \sim N(0,t)^3$$

$$EW(t)^3 = \int_{-\infty}^{\infty} x^3 \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} = 0$$

① $[X, X](t) < \infty$ $(\cancel{\epsilon \neq 0})$

then 1^{st} var $= 0$

② If 1^{st} var is finite.

then $QV = 0$

(cts process).

$$X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s).$$

Q: $E\,X(t)^2 \stackrel{?}{=}$ $[X, X]$.

① $E\left(\int_0^t \sigma(s)\,dW(s)\right)^2$ $M(t) = \int_0^t \sigma(s)\,dW(s).$

$$E\,M(t)^2 = E[M, M](t) + \underbrace{E M(0)^2}_{0}.$$

② $E\,X(t)^2 \stackrel{\text{need not}}{\neq} E[X, X](t).$

$$f(t) = \begin{cases} \xi_0 & t < t_1 \\ \xi_1 & t \geq t_1 \end{cases}$$

$$\int_0^t f(s)\,dW(s) = \begin{cases} \xi_0 W(t) & t < t_1 \end{cases}$$