Today:

- Martingales
- Ito Integrals
- Ito's Formula.

Q: If M_t is a martingale, does it have constant expectation?

i.e. is $\mathbb{E}[M_t] = c$ for every $t \geq 0$.

A: <u>YES!</u>. $s < t$.

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s.$$

Take $\mathbb{E}[\cdot]$ of both sides.

law of total expectation.

$$\text{RHS} = \mathbb{E}[M_s] = \mathbb{E}[\mathbb{E}[M_t \mid \mathcal{F}_s]] = \mathbb{E}[M_t].$$

take $s = 0$ $\quad \mathbb{E}[M_t] = \mathbb{E}[M_0] \rightsquigarrow$ constant.

Q: If $\mathbb{E}[M_t] = C \quad \forall t$, then is M_t a martingale?

A: <u>NO</u>

<u>Ex</u> Take $M_t = \dfrac{1+W_t^2}{1+t}$ → W_t is BM.

Then $\mathbb{E}[M_t] = \frac{1}{1+t}\left(1 + \mathbb{E}[W_t^2]\right) = \frac{1+t}{1+t} = 1$

take $s \leq t$

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\left[\frac{1+W_t^2}{1+t} \,\Big|\, \mathcal{F}_s\right] = \frac{1}{1+t}\left(1 + \mathbb{E}[W_t^2 \mid \mathcal{F}_s]\right).$$

$W_t \sim N(0,t) \Rightarrow \mathbb{E}[W_t^2] = t$ because t is the variance and W_t has mean 0.

$$\mathbb{E}[W_t^2 \mid \mathcal{F}_s] = \mathbb{E}[(\underbrace{W_t - W_s}_{a} + \underbrace{W_s}_{b})^2 \mid \mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^2 \mid \mathcal{F}_s] + 2\,\mathbb{E}[(W_t - W_s)W_s \mid \mathcal{F}_s] + \mathbb{E}[W_s^2 \mid \mathcal{F}_s]$$

$W_t - W_s \sim N(0, t-s)$

$$= \mathbb{E}[(W_t - W_s)^2] + 2W_s \mathbb{E}[W_t - W_s] + W_s^2$$

$$= t - s + 0 + W_s^2.$$

$$\boxed{\mathbb{E}[W_t^2 | \mathcal{F}_s] = W_s^2 + t - s.}$$

plug back in:

$$\mathbb{E}[M_t | \mathcal{F}_s] = \frac{1}{1+t}(1 + W_s^2 + t - s).$$

This ~~clearly depends~~ is clearly not M_s.

$$\left(M_s = \frac{1 + W_s^2}{1+s} \right).$$

so M_t is not a martingale, but it has constant expectation.

can rewrite equation in the box

$$\mathbb{E}[W_t^2 - t \mid \mathcal{F}_s] = W_s^2 - s.$$

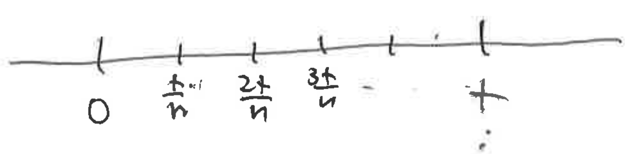
i.e. $N_t := W_t^2 - t$ is a martingale

Recall: If M_t is a martingale then

$M_t^2 - [M,M]_t$ is also a martingale...

Ito Integration.

$$I(t) := \int_0^t \Delta_s \, dW_s = \lim_{n \to \infty} \sum_{i=0}^{n-1} \left[\Delta_{\frac{ti}{n}} \left(W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \right) \right].$$



(i) $\boxed{\Delta \text{ must be adapted}}$

i.e. Δ_s is $\mathcal{F}_s$ meas for every s.

(ii) $\mathbb{E}\left[\int_0^t \Delta_s^2 \, ds\right] < \infty$. (typically in this class this will be satisfied).

I_t is a martingale.

$s < t$

i.e. $\mathbb{E}\left[\int_0^t \Delta_r dW_r \mid \mathcal{F}_s\right] = \int_0^s \Delta_r dW_r$.

We can now define Ito processes, which is a process X_t

$$X_t = X_0 + \underbrace{\int_0^t b_s ds}_{\text{drift}} + \underbrace{\int_0^s \sigma_s dW_s}_{\text{volatility}}. \quad (*)$$

Note b and σ can depend on time, W_t, X_t.

Shorthand Notation :

$$\begin{cases} dX_t = b_t\,dt + \sigma_t\,dW_t \\ X_0 = x_0. \end{cases}$$ as a shorthand

for $\textcircled{*}$,

Notice we omit X_0 in this notation, but it still matters

Ito Isometry :

$$\mathbb{E}\left[\left(\int_0^t \Delta_s\,dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t \Delta_s^2\,ds\right] < \infty.$$

Ito's Formula:

I want to apply some function f to X_t.

Do I still get a semimartingale (i.e. something of the form

$$d f(X_t) = \alpha_t dt + \gamma_t dW_t \text{)}$$

and if so how do I compute it?

If $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(t, X)$ is differentiable $\begin{cases} \text{once in time.} \\ \text{twice in } X \end{cases}$

then

$$(b_s ds + \sigma_s dW_s)$$

$$f(X_t) = f(X_0) + \int_0^t f_t(s, X_s)\, ds + \int_0^t f_x(s, X_s)\, dX_s.$$

$$+ \frac{1}{2}\int_0^t f_{xx}(s, X_s)\, d[X, X]_s.$$

what is QV of X?

If $dX_t = b_t\,dt + \sigma_t\,dW_t$.

then $d[X,X]_t = \sigma_t^2\,dt$

$$\Updownarrow$$

$$\left(\{X,X\}_t = \int_0^t \sigma_s^2\,ds \right)$$

So we can expand $f(X_t)$ as

$$f(X_t) = f(X_0) + \left[\underbrace{\int_0^t \left(f_t(s,X_s) + b_s f_x(s,X_s) + \tfrac{1}{2} f_{xx}(s,X_s)\,\sigma_s^2 \right) ds}_{\text{drift}} \right]$$

$$+ \underbrace{\int_0^t f_x(s,X_s)\,\sigma_s\,dW_s}_{\text{volatility}}.$$

Ex: Let's compute $\int_0^t W_s \, dW_s$.

Guess: what if it was $\int_0^t x\,dx = \frac{1}{2}t^2$.

So may it's $\frac{1}{2}W_t^2$?

Use Ito to get decomposition of $\frac{1}{2}W_t^2$.

Caution One cannot write $f(t,x) = \int_0^t \Delta_s \, dW_s$ and try to apply Ito. The RHS is not even differentiable.

define $f(t,x) = \frac{1}{2}x^2$

$$f_t(t,x) = 0 \qquad f_x(t,x) = x, \quad f_{xx}(t,x) = 1.$$

$$\frac{1}{2}W_t^2 = f(t,W_t) = f(0,W_0) + 0\,ds + \int_0^t W_s\,dW_s + \frac{1}{2}\int_0^t (1)\, \underbrace{d[W,W]_t}_{dt}.$$

$$= \int_0^t W_s\,dW_s + \frac{t}{2}.$$

$$\Rightarrow \int_0^t W_s \, dW_s = \cancel{\tfrac{1}{2}\left(W_t^2 - \tfrac{t}{2}\right)}. \quad \tfrac{1}{2}(W_t^2 - t).$$

FACT Stochastic integrals are martingales with mean 0.

Ex 2:

$$\begin{cases} dS_t = \mu S_t \, dt + \sigma S_t \, dW_t & (XX). \\ S_0 = s_0. \end{cases}$$

μ, σ are positive constants.

S given in this form is ~~the~~ a stochastic differential equation for S.

why? Because S appears in the RHS.

suppose $\sigma = 0$.

Then $dS_t = \mu S_t \, dt \iff \frac{dS_t}{dt} = \mu S_t$

Normal differential equation.

What's the solution?

written differently $f'(t) = \mu f$.

$\Rightarrow f(t) = f(0) e^{\mu t}$

i.e. $S_t = S_0 e^{\mu t}$

What about when $\sigma > 0$.

A guess would be $\boxed{S_t = S_0 e^{\mu t + \sigma W_t}}$ $\rightarrow$ will be wrong!

Want now to compute semimartingale decomposition of S_t.

define $f(t,x) = S_0 e^{\mu t + \sigma x}$.

$f_t(t,x) = \mu f(t,x)$. $\quad f_x(t,x) = \sigma f(t,x)$. $\quad f_{xx} = \sigma^2 f(t,x)$.

$$S_t = f(t, W_t) = f(0, W_0) + \int_0^t f_t(s, W_s)\,ds \cancel{+ \int_0^t \mu f(s, W_s)\,dx} + \int_0^t f_x(s, W_s)\,dW_s$$

$$+ \frac{1}{2}\int_0^t f_{xx}(s, W_s)\,ds.$$

$$= \cancel{S_0 + \int_0^t (\mu + \tfrac{\sigma^2}{2}) f(s, W}$$

$$= S_0 + \int_0^t \mu\, f(s, W_s)\, ds + \frac{\sigma^2}{2} \int_0^t f(s, W_s)\, ds + \sigma \int_0^t f(s, W_s)\, dW_s.$$

$$dS_t = (\mu + \boxed{\tfrac{\sigma^2}{2}}) S_t\, dt + \sigma S_t\, dW_t$$

it's not quite (xx).

<u>check</u> that $\cancel{S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma dW}}$

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

satisfies (xx)

This (S_t) is called Geometric Brownian Motion and is used as the stock price model in the Black Scholes Model.

Ex 3 $\quad dS_t = \mu S_t\,dt + \sigma S_t\,dW_t.$

~~find S_t^p & decomposition where.~~

find decomposition of S_t^p.

$$f(t,x) = x^p. \qquad f_x = p x^{p-1}, \quad f_{xx} = p(p-1)x^{p-2}$$

$$dS_t^p = df(t,S_t) = 0\,dt + pS_t^{p-1}\,dS_t + \frac{1}{2}p(p-1)S_t^{p-2}\,d[S,S]_t.$$

$$= pS_t^{p-1}\left(\mu S_t\,dt + \sigma S_t\,dW_t\right) + \frac{1}{2}p(p-1)S_t^{p-2}\left(\sigma^2 S_t^2\,dt\right).$$

$$= S_t^p\left(\left(p\mu + \frac{1}{2}p(p-1)\sigma^2\right)dt + p\sigma\,dW_t\right).$$

For what values of p is S_t^p a martingale?

Fact. If $dX_t = b_t\,dt + \sigma_t\,dW_t$

then X_t is a martingale $\iff b_t = 0$.

We just need to find p s.t.

$$p\mu + \frac{1}{2}p(p-1)\sigma^2 = 0.$$

$$p\left(\mu + \frac{1}{2}(p-1)\sigma^2\right) = 0.$$

$$\boxed{p=0}$$

or $\frac{1}{2}(p-1) = -\frac{\mu}{\sigma^2} \iff \boxed{p = 1 - \frac{2\mu}{\sigma^2}}$

To find the decomposition ~~you~~ of $f(t, X_t)$.

(i) ~~find~~ take function $f(t,x)$ s.~~uch that.~~ and compute it's derivatives normally.

(ii) plug in X_t for x everywhere in Ito's formula.

Ex 4 $X_t = te^{3W_t}$ Find decomposition.

$$f(t,x) = te^{3x}$$

$$f_t(t,x) = e^{3x} \qquad f_x(t,x) = 3te^{3x} \qquad f_{xx}(t,x) = 9te^{3x}.$$

$$X_t = f(t,W_t) = \underset{0}{\underset{\shortparallel}{f(0,0)}} + \int_0^t e^{3W_s}\,ds + 3\int_0^t se^{3W_s}\,dW_s + \frac{9}{2}\int_0^t se^{3W_s}\,ds.$$

$$= \int_0^t \left(e^{3W_s}\left(1+\tfrac{9}{2}s\right)\right)ds + 3\int_0^t se^{3W_s}\,dW_s.$$

Q: If $\cancel{Z}$ $Z = X + Y$ are Ito processes

is $[Z,Z]_t = [X,X]_t + [Y,Y]_t$?

A: No! Be careful.

$$dX_t = b_t\,dt + \sigma_t\,dW_t$$

$$dY_t = \alpha_t\,dt + \gamma_t\,dW_t.$$

$$dZ_t = (b_t + \alpha_t)dt + (\sigma_t + \gamma_t)\,dW_t.$$

$$d[Z,Z]_t = (\sigma_t + \gamma_t)^2\,dt$$

$$d[Y,Y]_t = \gamma_t^2\,dt$$ not generally equal!!

$$d[X,X]_t = \sigma_t^2\,dt$$

Common Mistakes on last year's:

1. $W_t \perp\!\!\!\perp W_s$ instead of $W_s \perp\!\!\!\perp W_t - W_s$. $t \geq s$.

FALSE

- Incorrect use of Ito's formula.

- Not explaining steps. → Important!

- $\mathbb{E}[e^X] \cancel{=} e^{\mathbb{E}[X]}$.

↳ Don't forget old rules for Expectations etc...

$\mathbb{E}[e^{tZ}] = e^{\frac{t^2}{2}}$ $N(0,1)$