① MIDTERM $\longrightarrow$ In class (Closed book).

Last time: Itô Integral.

$W \longrightarrow$ Brownian Motion.

$D \longrightarrow$ some adapted process. ($D(t)$ is meas wrt $\mathcal{F}_t$).

Constructed $\int_0^T D(t)\, dW(t) = I(T)$ (Itô integral).

$\hookrightarrow$ ⓪ $I(T)$ is a cts adapted process $(s<t)$.

① $I(T)$ is a mg $\left(E(I(t) \mid \mathcal{F}_s) = I(s) \right)$.

② $[I, I](T) = \int_0^T D(t)^2\, dt$

Semi Martingales:

Processes of the form $X(t) = \underbrace{X(0)}_{\mathcal{F}_0 \text{ meas. R.V.}} + \underbrace{B(t)}_{\text{finte } 1^{st} \text{ varition cts adapted.}} + \underbrace{M(t)}_{\text{cts, } \underline{\text{mg}}}$

(also want $B(0) = M(0) = 0$).

Typically $\quad B(t) = \int_0^t b(s)\, ds \quad$ (Rieman Int).

& $\quad M(t) = \int_0^t \sigma(s)\, dW(s) \quad$ (Itô integral).

b & σ are adapted processes.

Writing $X = X(0) + \underbrace{B(t)}_{\text{finite } 1^{st} \text{ var.}} + \underbrace{M(t)}_{\text{mg}}$ is called the Itô decompostion

(bounded variation) $\longrightarrow$ B.V.

If $B(t) = \int_0^t b(s)\,ds$ & $M(t) = \int_0^t \sigma(s)\,dW(s)$.

then $\underbrace{X(t) - X(0)}_{\int_0^t dX(s)} = \underbrace{\int_0^t b(s)\,ds}_{\text{Riem Int}} + \underbrace{\int_0^t \sigma(s)\,dW(s)}_{\text{Itô int}}$.

Short hand notation: $\boxed{dX(t) = b(\underbrace{t}_{\text{dummy var.}})\,dt + \sigma(t)\,dW(t)}$

Compute $[X, X](T)$, where. $X = X(0) + B + M$.

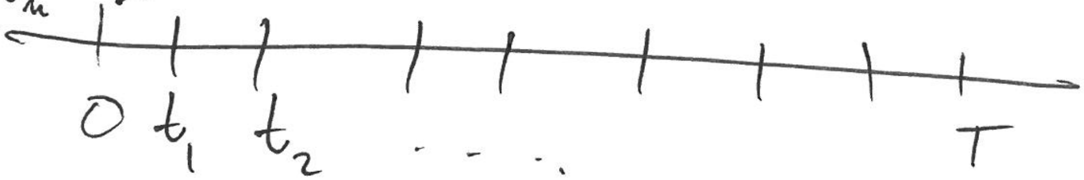
& $B = \int_0^t b(s)\,ds$, $M(t) = \int_0^t \sigma(s)\,dW(s)$

Guess. $[X, X](T) = \int_0^T \sigma^2(t)\,dt + \cancel{\int_0^T b(t)^2\,dt}$

<u>Prop:</u> $[X, X](T) = \int_0^T \sigma(t)^2\,dt$

$$\left(\Longleftrightarrow d[X,X](t) = \sigma^2(t)\,dt \right).$$

Intuition: $P = \{0 = t_0 < t_1 < \cdots t_n = T\}$.



$$\Delta_i B = B(t_{i+1}) - B(t_i) = \int_{t_i}^{t_{i+1}} b(s)\, ds.$$

$$\Delta_i M = M(t_{i+1}) - M(t_i). = \int_{t_i}^{t_{i+1}} \sigma(s)\, dW(s).$$

$$\Delta_i X = X(t_{i+1}) - X(t_i).$$

$$[X, X](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)^2$$

$$\sum_{i=0}^{n-1} (\Delta_i X)^2 = \underbrace{\sum_{i=0}^{n-1} (\Delta_i M)^2}_{\xrightarrow{\|P\| \to 0}\ [M,M](T) = \int_0^T \sigma(t)^2\, dt} + \underbrace{\sum_{i=0}^{n-1} (\Delta_i B)^2}_{\text{Claim \&}\ \xrightarrow{\|P\| \to 0}\ 0} + 2 \underbrace{\sum_{i=0}^{n-1} (\Delta_i B)(\Delta_i M)}_{\longrightarrow\ 0}$$

① $$\sum_{i=0}^{n-1} (\Delta_i B)^2 \leq \left(\max_i |\Delta B_i|\right) \cdot \sum_{i=0}^{n-1} |\Delta_i B|.$$

$$\leq \left(\max_i \int_{t_i}^{t_{i+1}} |b(s)|\, ds.\right) \cdot \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |b(s)|\, ds.$$

$$\leq \max_i |b| \cdot (t_{i+1} - t_i) \int_0^T |b(s)|\, ds.$$

$$\leq \left(\max_{[0,T]} |b|.\right). \|P\| \cdot \int_0^T |b(s)|\, ds. \xrightarrow{\|P\| \to 0.} 0$$

③ $$\sum_{i=0}^{n-1} (\Delta_i B)(\Delta_i M) \leq \underbrace{\left(\sum_{i=0}^{n-1} (\Delta_i B)^2\right)^{1/2}}_{\xrightarrow{\|P\| \to 0}\ 0} \underbrace{\left(\sum_{i=0}^{n-1} (\Delta_i M)^2\right)^{1/2}}_{\xrightarrow{\|P\| \to 0}\ [M,M](T)}.$$

(Cauchy Schwarz).

$\Rightarrow$ If $dX = b\,dt + \sigma\,dW$.

then $[X, X](T) = \int_0^T \sigma(t)^2\,dt$

$\left(\Leftrightarrow \ d[X,X](t) = \sigma(t)^2\,dt\right)$.

Notation: Let D be some adapted process. (# shares on an asset).

$X \to$ a semi mg. $\quad dX = b\,dt + \sigma\,dW$.

($X \to$ spot price of an asset).

reg Riemann Int (not mg, 0 QV).

Wealth:
$$\int_0^T D(t)\,dX(t) \overset{def}{=} \overbrace{\int_0^T D(t)\,b(t)\,dt}^{\text{reg Riemann Int (not mg, 0 QV)}} + \cancel{\int_0^T \sigma\,dt}$$
$$+ \underbrace{\int_0^T D(t)\,\sigma(t)\,dW(t)}_{\text{Itô int (mg, finite QV)}}.$$

Itô formula:

⓪ Chain rule: Say f is a diffb fn of t & x.

Suppose X is a diff fn of t.

Chain rule $\quad \partial_t f(t, X(t)) = \frac{d}{dt} f(t, X(t))$.

$$= \partial_t f(t, X(t)) \cdot \frac{dt}{dt} + \partial_x f(t, X(t)) \cdot \partial_t X(t).$$

$$\Rightarrow f(T, X(T)) - f(0, X(0)) = \int_0^T \partial_t f(t, X(t))\, dt + \int_0^T \partial_x f(t, X(t))\, dX(t)$$

$$\Leftrightarrow \quad d f(t, X(t)) = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t).$$

① Itô formula: $f \to$ some diff fn of t & x.

$X \to$ semi mg.

$$dX = \cancel{\sigma(t)\,dt +}\; b(t)\,dt + \sigma(t)\,dW(t).$$

(NEVER a diff fn of t).

Q: What is $d f(t, X(t))$.

Ans: ~~Guess~~:

$$d f(t, X(t)) = \partial_t f(t, X(t))\,dt + \partial_x f(t, X(t))\,dX(t) + \underbrace{\frac{1}{2}\partial_x^2 f(t, X(t))\,d[X, X](t)}_{\text{extra} \to \text{Itô correction}}.$$

Itô formula.

(Itô Doeblin formula).

Prop: (Itô's formula).

① f is a fn of t & x.

f is once diff wrt t

f is TWICE diff wrt x.

② b & σ are adapted processes. $\left(E\int_0^t b(s)^2 ds < \infty \right.$

$\left. E\int_0^t \sigma(s)^2 ds < \infty \right)$.

Let $X(t) = X(0) + \int_0^t b(s)\, ds + \int_0^t \sigma(s)\, dW(s)$.

③ Then $f(T, X(T)) - f(0, X(0)) = \int_0^T \partial_t f(t, X(t))\, dt +$

$+ \int_0^T \partial_x f(t, X(t))\, dX(t) + \frac{1}{2}\int_0^T \partial_x^2 f(t, X(t))\, d\underbrace{[X,X](t)}_{\sigma^2(t)\, dt}$

Itô + Riemann.

Intuition: ~~Try~~ ~~f(t)~~ For simplicity assume $f = f(x)$ (no t dependence).

$$X(t) = W(t) \qquad (b=0,\ \sigma=1).$$

$$\cancel{f'(t) =} \ f'(x) = \frac{df}{dx} = \partial_x f.$$

Itô:
$$\left.\begin{aligned} d\,f(W(t)) &= 0 + f'(W(t))\,dW(t) \\ &\quad + \frac{1}{2} f''(W(t))\,dt \end{aligned}\right\} (*)$$

Will try & show $(*)$.

$$P = \{0 = t_0 < t_1 < \cdots \; t_n = T\}.$$

$$\boxed{f(a+h) - f(a) \approx h f'(a) + \frac{h^2}{2} f''(a)}$$

$$f(W(T)) - f(W(0)) = \sum_{i=0}^{n-1} f(W(t_{i+1})) - f(W(t_i)).$$

$$\approx \sum_{i=0}^{n-1} f'(W(t_i))\left(W(t_{i+1}) - W(t_i)\right) + \frac{f''(W(t_i))}{2}\left(W(t_{i+1}) - W(t_i)\right)^2$$

$+$ higher order terms.

$$\approx \underbrace{\sum_{i=0}^{n-1} f'(W(t_i))\, \Delta_i W}_{\substack{\downarrow\ \|P\| \to 0 \\ \int_0^T f'(W(t))\, dW(t)}} + \underbrace{\sum_{i=0}^{n-1} \frac{f''(W(t_i))}{2} (\Delta_i W)^2}_{\substack{\downarrow\ \|P\| \to 0 \\ \frac{1}{2}\int_0^T f''(W(t))\, dt}} + \underbrace{H.O.T.}_{\downarrow\ 0.}$$

Eg: Compute QV of $W(t)^2$

Guess: $\cancel{t^2 ?}$

Let $f(t, x) = x^2$; $\partial_t f = 0$, $\partial_x f = 2x$, $\partial_x^2 f = 2$.

$X(t) = W(t)$. $dX = dW$. $d[X,X] = dt$.

By Itô $d f(W(t)) = d(W(t)^2)$

$$= \partial_t f \, dt + \partial_x f \, dX + \frac{1}{2} \partial_x^2 f \, d[X,X].$$

$$= 0 + 2W(t) \, dW(t) + \frac{1}{2} \cdot 2 \cdot dt.$$

$$= 2W(t) \, dW(t) + dt.$$

$$\Rightarrow W(T)^2 = \int_0^T 2W(t)\,dW(t) + \int_0^T dt.$$

$$\boxed{\Rightarrow [W^2, W^2](T) = \int_0^T 4W(t)^2\,dt}$$

(not t^2).

Let $X(t) = t \sin W(t)$

Q: Is $X(t)^2 - [X,X]$ a mg? (Guess: Yes).

Know: If M is a mg, then $M(t)^2 - [M,M](t)$ is a mg.

$X \to$ may not be a mg.

Above prop may not be true for non mg.

Check: Strategy: ① ~~Compute $E(X$~~ Let $N(t) = X(t)^2 - [X,X](t)$.

compute $E(N(t) \mid \mathcal{F}_s)$.

Better strat: ② Itô formula.

$$\text{Compute } dN = \underbrace{(\ )}_{\substack{\text{not a mg} \\ \text{unless it is } 0!}} dt + \underbrace{(\ )}_{\text{mg}} dW$$

If the dt terms vanish, N is a mg.

Otherwise N is not a mg.

① Let $f(t, x) = t \sin x$.

$X(t) = f(t, W(t))$.

$\partial_t f = \sin x$

$\partial_x f = t \cos x$

$\partial_x^2 f = -t \sin x$.

$$dX = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, dt$$

$$= \sin W(t) \, dt + t \cos W(t) \, dW(t) - \frac{1}{2} t \sin W(t) \, dt$$

$$= \sin W(t) \left(1 - \frac{t}{2}\right) dt + t \cos W(t) \, dW(t)$$

$$\boxed{\Rightarrow d[X, X](t) = t^2 \cos^2 W(t) \, dt}$$

*

Let $N = X^2 - [X,X]$. & compute dN.

~~dN~~ Itô $\Rightarrow$ $dN = d(X^2) - d[X,X]$. $\cancel{(*)}$.

$$= 2X\,dX + \frac{1}{2}\cdot 2\,d[X,X] - d[X,X].$$

$$= 2X\,dX + \cancel{d[X,X]} - \cancel{d[X,X]}.$$

$$= 2X\left(\sin W(t)\left(1-\frac{t}{2}\right)\right)dt \leftarrow \text{not mg!!.}$$

$$+ \underbrace{2Xt\cos(W(t))\,dW(t)}_{\text{mg}}$$

$\Rightarrow X^2 - [X,X]$ is NOT a mg!!.

If ~~X~~ $dX = b\,dt + \sigma\,dW$

Q: If $X^2 - [X,X]$ is a mg

must X be a mg?

Yes: $N = X^2 - [X,X]$.

$$dN = 2X\,dX + \frac{1}{2}\cdot 2\,\cancel{d[X,X]} - \cancel{d[X,X]}.$$

$$= 2X\,b\,dt + 2X\sigma\,dW$$

If N is a mg, then $2Xb = 0 \Rightarrow b = 0$

$\Rightarrow X$ is a mg.

Uniqueness of S. mg decomp:
(IOU).

Claim: If $X = X(0) + B_1 + M_1$ $\quad$ (B_i — BV

& $X = X(0) + B_2 + M_2$ $\quad$ $M_i \to$ mg.

Then: $B_1 = B_2$ & $M_1 = M_2$.

Claim ⇒ that ~~a~~ B is a mg $\Leftrightarrow$ $B = 0$.
(B is cts adapted, ~~finite~~ BV, $B(0) = 0$).

Pf: Suppose B is a mg

$\because$ B is BV $\Rightarrow [B, B] = 0$

$\because$ B is a mg, $B^2 - [B, B]$ is mg

$\Rightarrow B^2$ is a mg.

$\Rightarrow E B^2(t) = E B^2(0) = 0$

$\Rightarrow B(t) = 0.$