- Midterm next Thursday

- Markov Processes $\not\Leftrightarrow$ Martingale Processes

Ex 1. W is B.M. wrt $\{\mathcal{F}_t\}_{t\ge 0}$ 1) B.M is Markov

(1) $\{e^{W(t)}\}_{t\ge 0}$ a Markov? Martingale?

$$t \ge s \ge 0,\quad \mathbb{E}[e^{W(t)} \mid \mathcal{F}_s] = \mathbb{E}[e^{\underbrace{W(t)-W(s)}_{\perp \mathcal{F}_s} + W(s)} \mid \mathcal{F}_s]$$

$$= e^{W(s)}\, \mathbb{E}[e^{W(t)-W(s)}]$$

Recall. $X \sim N(\mu, \sigma^2)$

$$\mathbb{E}[e^{Xt}] = \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)$$

MGF

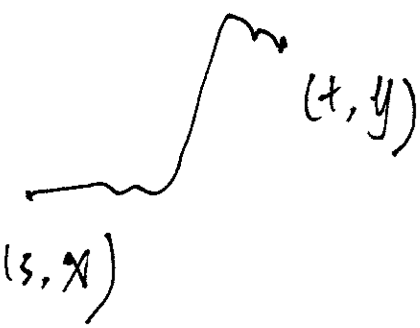
t - constant.

$$\mathbb{E}\left[e^{W(t)-W(s)}\right] = \exp\left(\tfrac{1}{2}(t-s)\right)$$

$$\mathbb{E}\left[e^{W(t)} \mid \mathcal{F}_s\right] = e^{W(s)+\frac{1}{2}t-\frac{1}{2}s} = e^{\frac{1}{2}t}\, e^{W(s)-\frac{1}{2}s}.$$

$\left\{e^{W(t)-\frac{1}{2}t}\right\}_{t\geq 0}$ is a m'gale

- m'gale wrt some $\mathbb{P}$



(2) $\max_{0 \le s \le t} W(s) =: M(t)$ $\quad \{M(t)\}_{t \ge 0}$ is a m'gale?

No. For $0 \le s \le t$, $\quad M(s) \le M(t)$, a.s

$$M(s) \le E[M(t) \mid \mathcal{F}_s] \quad a.s$$

$\{M(t)\}_{t \ge 0}$ is Markov? No.

$\{(M(t), W(t))\}_{t \ge 0}$, is Markov.

(3). $X_0 = 0$, $\Delta(t)$ is a $\mathcal{F}_t$-measurable processes.

$X(t) = \int_0^t \Delta(u)\, dW_u + \int_0^t \theta(u)\, du$, is $\{X(t)\}_{t \ge 0}$ a m'gale?

$$dX(t) = \Delta(t)\, dW(t) + \theta(t)\, dt$$

unless $\theta(t) \equiv 0$ $\forall t \ge 0$, X is not a m'gale

Assign2 4(b) ④

$$E[(e^{X+Y}-K)^+ | X] = \int_{-\infty}^{+\infty} (e^{X+Y}-K)^+ \, p(Y|X) \, dy$$

$Y \geq$ lowerbound

$$= \int_{\text{lowerbound}}^{+\infty} (e^{X+Y}-K) \, p(Y|X) \, dy$$

Result is in the form of ~~cdf~~ standard Gaussian cdf

$X \perp\!\!\!\perp Y$

$p(Y|X) = p(Y)$.

Ex 2. $\{W(t)\}_{t\geq 0}$ be a B.M. $\mathbb{P}$, wrt $\{\mathcal{F}_t\}_{t\geq 0}$: (5)

(1) $M(t) := \mathbb{E}\left[\int_0^T W(u)\,du \,\middle|\, \mathcal{F}_t\right]$,

1) $t \geq T \geq 0$ $\quad M(t) = \int_0^T W(u)\,du$

2) $T \geq t \geq 0$ Complete the integral / Separate.

$$\mathbb{E}\left[\int_0^T W(u)du \,\middle|\, \mathcal{F}_t\right] = \underbrace{\int_0^t W(u)\,du}_{\text{past}} + \mathbb{E}\left[\underbrace{\int_t^T W(u)\,du}_{\text{future}} \,\middle|\, \mathcal{F}_t\right]$$

$$\mathbb{E}\left[\int_t^T W(u)\,du \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \underbrace{W(u) - W(t)}_{0} + W(t)\,du \,\middle|\, \mathcal{F}_t\right]$$

b/c $\mathbb{E}[W(u) - W(t) \mid \mathcal{F}_t] = 0$

$$= \int_t^T W(t)\,du = W(t)\,(T-t).$$

$t \geq s \geq 0$

summary $\mathbb{E}[f(W(t)) \mid \mathcal{F}_s] = \mathbb{E}[f(W(t)-W(s)+W(s)) \mid \mathcal{F}_s$

Then apply independence lemma on $(W(t)-W(s), W(s))$

Ex (2) $M(t) := \mathbb{E}[W^2(T) \mid \mathcal{F}_t]$

$$= \mathbb{E}[(W(T)-W(t)+W(t))^2 \mid \mathcal{F}_t]$$

$$= \mathbb{E}[(W(T)-W(t))^2 + (W(t))^2 + 2(W(T)-W(t))W(t) \mid \mathcal{F}_t]$$

(3) $M(t) := \mathbb{E}[W^n(T) \mid \mathcal{F}_t]$

Ex3. X, has pdf $p(X) > 0$ under prob measure $\mathbb{P}$, $\mathcal{F}$

$Y = g(X)$, g is strictly increasing, $g(y) \downarrow -\infty,\ y \downarrow -\infty$

$g(y) \uparrow +\infty,\ y \uparrow +\infty$

$h \geq 0$, $\underline{\int_{-\infty}^{+\infty} h(y)\,dy = 1}$

Goal: choose prob measure $\widehat{\mathbb{P}}$ s.t under $\widehat{\mathbb{P}}$, Y has density h.

$$Z = \frac{h(g(X))\, g'(X)}{p(X)}$$

(i) $\mathbb{E}^{\mathbb{P}}[Z] = 1$

$$\mathbb{E}^{\mathbb{P}}[Z] = \int_{-\infty}^{+\infty} \frac{h(g(x))\, g'(x)}{\cancel{p(x)}} \cdot \cancel{p(x)}\, dx$$

$$= \int_{-\infty}^{+\infty} h(g(x)) \cdot g'(x)\, dx$$

change of variables $g(x) = y$

$dy = g'(x)\,dx$

$$= \int_{-\infty}^{+\infty} h(y)\, dy = 1$$

(ii) Create the New measure $\hat{\mathbb{P}}$

1) $\hat{\mathbb{P}}$ is prob. measure ✓

2) under $\hat{\mathbb{P}}$, Y has density h

$$\forall A \in \mathcal{F}, \quad \hat{\mathbb{P}}[A] = \int_A Z \, d\mathbb{P}$$

original prob measure

1) $\forall A \in \mathcal{F}$, $\hat{\mathbb{P}}[A] \geq 0$. ✓

check $h(g(X)) \geq 0$, $g'(X) > 0$ $\quad p(X) > 0$

2) $\hat{\mathbb{P}}[\Omega] = 1$

check: $\hat{\mathbb{P}}[\Omega] = \int_{-\infty}^{+\infty} Z \, p(X) \, dx = E^{\mathbb{P}}[Z] = 1$

3) $\hat{\mathbb{P}}[Y \leq y] = \int_{-\infty}^{+\infty} Z \, p(X) \cdot \mathbb{1}_{[Y \leq y]} \, dX$

$h(Y)$

$$= \int_{-\infty}^{+\infty} \frac{h(g(X)) \, g'(X)}{\cancel{p(X)}} \cdot \cancel{p(X)} \, \mathbb{1}_{[Y \leq y]} \, dX$$

dY

$$\hat{\mathbb{P}}[Y \leq y] \overset{Y = g(X)}{=\!=\!=} \int_{-\infty}^{+\infty} \mathbb{1}_{[Y \leq y]} \cdot h(Y)\, dY$$

$$= \int_{-\infty}^{y} h(Y)\, dY.$$

Change of Measure

Z - rule