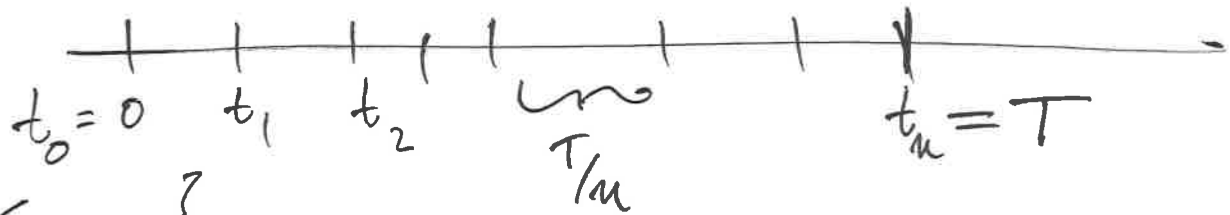
Last time: $(s \le t)$

Mg $\longrightarrow$ Fair Game. $E(M(t) \mid \mathcal{F}_s) = M(s)$

Quadratic Variation:



$P = \{0 = t_0 < t_1 < \cdots\}$.

$\|P\| = \max t_{i+1} - t_i$

$W \longrightarrow$ B.M. $\quad \Delta_i W = W(t_{i+1}) - W(t_i)$

$n$ terms of size $\sqrt{\frac{T}{n}} \approx \sqrt{nT} \to \infty$

1st variation: $\displaystyle\lim_{\|P\| \to 0} \sum_{i=0}^{n-1} |\Delta_i W| \longrightarrow +\infty$ almost surely.

$N(0, t_{i+1} - t_i)$

$|\Delta_i W| \approx \sqrt{t_{i+1} - t_i} \approx \frac{\sqrt{T}}{\sqrt{n}}$

Quadratic Variation:

$X \to$ some process. $P = \{0 = t_0 < t_1 \cdots t_n = T\}$.

$$[X, X](T) \stackrel{\text{def}}{=\!=} \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)^2$$

Last time: $[W, W](T) = T$ (Q.V of BM $= T$).

Q: $W \longrightarrow$ is a mg.

Q: $W^2 \longrightarrow$ is not a mg

Compute $E(W(t)^2 \mid \mathcal{F}_s) = E((W(t) - W(s) + W(s))^2 \mid \mathcal{F}_s)$.

$$= E((W(t) - W(s))^2 + W(s)^2 + 2(W(t) - W(s))W(s) \mid \mathcal{F}_s).$$

$$= t-s + W(s)^2 + 2W(s)\underbrace{E\left(W(t)-W(s) \mid \mathcal{F}_s\right)}_{0}.$$

$$= W(s)^2 + t - s$$

$$E\left(W(t)^2 \mid \mathcal{F}_s\right) = W(s)^2 + t - s. \Rightarrow W^2 \text{ is NOT a mg}$$

Let $M(t) = W(t)^2 - t$. Q: Is $M$ a mg.

$$E\left(M(t) \mid \mathcal{F}_s\right) = E\left(W(t)^2 - t \mid \mathcal{F}_s\right) = W(s)^2 + t - s - t$$
$$= M(s)$$

$\Rightarrow W(t)^2 - t$ is a mg.

$\Rightarrow W(t)^2 - [W,W](t)$ is a mg!

<u>Theorem</u>: (1) Let $M$ be a cts mg wrt the filt $\{\mathcal{F}_t\}$.

Then $EM(t)^2 < \infty \iff E[M,M](t) < \infty$

& in this case the process $M(t)^2 - [M,M](t)$ is also a mg.

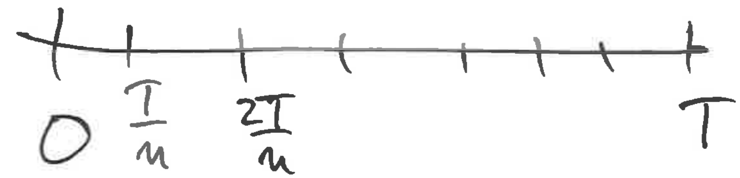
$$\Big[ \Rightarrow E\big(M(t)^2 - [M,M](t)\big) = E\big(M(0)^2 - \underbrace{[M,M](0)}_{0}\big).$$

$$\Rightarrow E\big(M(t)^2 - M(0)^2\big) = E[M,M](t). \Big]$$

(2) Conversely, if $A(t)$ is any cts increasing adapted process such that $A(0)=0$ & $M^2(t) - A(t)$ is a mg then $A(t) = [M,M](t)$.

Intution: If $X$ has finite $1^{st}$ variation.

$$\lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1}}_{n \text{ terms.}} \underbrace{|\Delta_i X|} < \infty.$$




Expect $|\Delta_i X| \approx \frac{1}{n}$

Expect $\left| X\left( \frac{(i+1)T}{n} \right) - X\left( \frac{iT}{n} \right) \right| \approx \frac{1}{n}.$

i.e. Expect $|X(t+\delta t) - X(t)| \approx \delta t$

If $X$ has finite Quadratic variation.

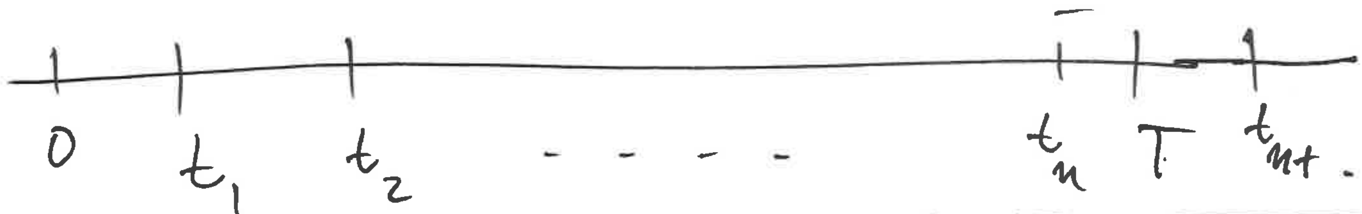
$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (\Delta_i X)^2 < \infty \Rightarrow (\Delta_i X)^2 \approx \frac{1}{n}$$

Expect $\left(X(t+\delta t) - X(t)\right)^{\cancel{2}} \approx \sqrt{\delta t}$

Construct Ito Integrals. $W \to$ BM. $\{\mathcal{F}_t\} \to$ Brownian Filt.

$\boxed{D(t) \to \text{adapted process.}}$ (# shares held at time $t$ on an asset with spot price $W(t)$).

$P = \{0 = t_0 < \cdots\}$.




Only trade at times $t_i$

$$\boxed{W(t_{i+1}) - W(t_i) \overset{\text{def}}{=} \Delta_i W}$$

Let
$$I_P(T) \overset{\text{def}}{=} \sum_{i=0}^{n-1} D(t_i)(\Delta_i W) + D(t_n)\big(W(T) - W(t_n)\big).$$

(cumulative wealth up to time $T$).

Lemma: ① $E\, I_p(T)^2 = E\left[\sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1}-t_i)\right]$

$+ \; E\, D(t_n)^2 (T-t_n) \quad (\text{if } T \in [t_n, t_{n+1}))$

② $I_p(t)$ is a martingale.

③ $[I_p, I_p](T) = \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1}-t_i) + D(t_n)^2 (T-t_n)$

if $T \in [t_n, t_{n+1})$.

Check #1: Assume $\boxed{T = t_n}$.

N.T.S $\quad E\, I_p(T)^2 = \underbrace{E \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i)}_{\text{RHS}}.$

$$I_p(T) = \sum_{i=0}^{n-1} D_i(t_i)\, \Delta_i W$$

$$E\, I_p(T)^2 = \underbrace{E \sum_{i=0}^{n-1} D(t_i)^2 (\Delta_i W)^2}_{a} + \underbrace{2E \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} D(t_i) D(t_j) (\Delta_i W)(\Delta_j W)}_{b}.$$

Compute $E$ a:
$$E\, D(t_i)^2 (\Delta_i W)^2 = E\, D(t_i)^2 (W(t_{i+1}) - W(t_i))^2$$

$$=$$

Note $W(t_{i+1}) - W(t_i)$ is ind of $\mathcal{F}_{t_i}$.

$D(t_i)$ is meas wrt $\mathcal{F}_{t_i}$

$\Rightarrow \left(W(t_{i+1}) - W(t_i)\right)^2$ is ind of $D(t_i)^2$.

$$\Rightarrow E\, D(t_i)^2 (\Delta_i W)^2 \underset{\text{ind}}{=} \left(E\, D(t_i)^2\right) \Big(\underbrace{E (\Delta_i W)^2}_{t_{i+1} - t_i}\Big).$$

$$= E\, D(t_i)^2 (t_{i+1} - t_i).$$

$$\Rightarrow \underbrace{E \sum_{i=0}^{n-1} D(t_i)^2 (\Delta_i W)^2}_{ⓐ} = \underbrace{E \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i)}_{RHS}.$$

To finish N.T.S. (b) $= 0$.

For terms in (b) $\underline{i} < j$. $\Rightarrow t_i < t_j$ & $t_{i+1} \leq t_j$

$$E\, D(t_i) D(t_j) (\Delta_i W)(\Delta_j W) = \underline{E}\, E\left( D(t_i) D(t_j)\, \Delta_i W\, \Delta_j W \,\middle|\, \mathcal{F}_{t_j} \right)$$

$$= E\left[ D(t_i) D(t_j) (\Delta_i W)\, \underbrace{E\left( \Delta_j W \,\middle|\, \mathcal{F}_{t_j} \right)}_{0} \right].$$

($\because D(t_i), D(t_j), \Delta_i W$ are meas wrt $\mathcal{F}_{t_j}$).

$$= 0$$

//.

You check : $I_p(T)$ is a mg. (tower prop).

To check 3.

Let $A(T) = \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + D(t_n)^2 (T - t_n)$.

Check (tower prop) $I_p(T)^2 - A(T)$ is a mg.

$\Rightarrow A(T) = [I_p, I_p](T)$. (You check).

Note : $$\lim_{\|P\| \to 0} [I_p, I_p](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i)$$

$$= \int_0^T D(t)^2 \, dt$$

Itô : As $\|P\| \to 0$ $\{ I_p$ is a mg.

& as $\|P\| \to 0$, $[\cancel{P} I_p, I_p](T) \longrightarrow \int_0^T D(t)^2 \, dt$.

$\Rightarrow$ The processes $I_p$'s themselves converge as. mg's.

Itô integral: $$\lim_{\|P\| \to 0} I_p.$$

Theorem (Itô): If $\int_0^T D(t)^2 dt < \infty$ almost surely,

then $\lim_{\|P\| \to 0} I_P$ exists. & ~~is a mg.~~

Let $I(T) = \lim_{\|P\| \to 0} I_P(T)$. is called the Itô integral.

Notation: $I(T) = \int_0^T D(\underset{\text{in}}{t})\, dW(\underset{\text{in}}{t})$

dummy variable

If $E\int_0^T D(t)^2 dt < \infty$, then $I$ is a martingale.

in this case $[I, I](t) = \int_0^T D(t)^2 dt$.

**NOTE:** Require $D$ to be an <u>adapted process</u>.

**Cor:** If $E\int_0^T D(t)^2\,dt < \infty$, then

$$\underbrace{E\left(\int_0^T D(t)\,dW(t)\right)^2}_{I} = E\int_0^T D(t)^2\,dt.$$

(Itô isometry).

$$\boxed{\text{Goal: Itô formula. (Stochastic version of chain rule).}}$$

Prop of Itô integrals:

① Say $D_1$ & $D_2$ are 2 adapted processes.

$$\int_0^T (D_1(t) + D_2(t))\, dW(t) = \int_0^T D_1(t)\, dW(t) + \int_0^T D_2(t)\, dW(t).$$

② $\alpha \in \mathbb{R}$, $\displaystyle\int_0^T \alpha D(t)\, dW(t) = \alpha \int_0^T D(t)\, dW(t)$.

~~③ If $D(t) \geq 0$ then $\displaystyle\int_0^T D(t)\, dW(t) \geq 0$~~

NO!!