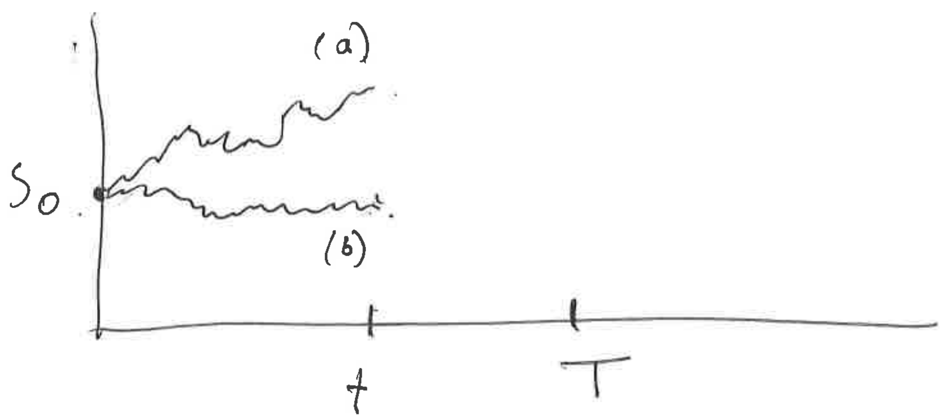
Today : — Conditional EXP.

— Brownian Motion

— Martingales.

(2) intuitively means the conditional expectation $\mathbb{E}[X \mid \mathcal{F}]$ is on average our best estimate of X incorporating only the information given by $\mathcal{F}$. (1).



$\{S_T > K\}$.
event of interest.

$X_t \rightarrow$ ~~payoff of~~ value some derivative that expires at call T.

clearly at time t the ~~price~~ value ~~of~~ X_t will be different than X_0.

<u>Ex</u> $(\Omega, \mathcal{G}, \mathbb{P})$. $A, B \in \mathcal{G}$ $\mathbb{P}(B) > 0$.

$$\mathcal{F} = \sigma(\mathbb{1}_B) = \{\phi, B, B^c, \Omega\}.$$

<u>Claim</u> $\mathbb{E}[\mathbb{1}_A | \sigma(\mathbb{1}_B)] = \underbrace{\mathbb{P}(A|B)\, \mathbb{1}_B + \mathbb{P}(A|B^c)\, \mathbb{1}_{B^c}}_{X}$.

$$\left(\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \right).$$

<u>Verification</u> (1) Need X to be $\sigma(\mathbb{1}_B)$ meas.

$$X(\omega) = \begin{cases} \mathbb{P}(A|B) & ; \omega \in B. \\ \mathbb{P}(A|B^c) & ; \omega \in B^c. \end{cases}$$

Look at sets of the form $\{\omega : X(\omega) < a\}$.

$c = \min\{\mathbb{P}(A|B),\ \mathbb{P}(A|B^c)\}$, $d = \max\{\mathbb{P}(A|B),\ \mathbb{P}(A|B^c)\}$

if $a \le c$:

$$\{\omega : X(\omega) < a\} = \emptyset \in \mathcal{F}.$$

if $c < a \le d$. $$\{\omega : X(\omega) < a\} = \begin{cases} B & : \ \mathbb{P}(A|B) = c. \quad \cancel{\mathbb{P}(A|B) < \mathbb{P}(A|B^c)} \\ B^c & : \ \mathbb{P}(A|B^c) = c. \end{cases} \in \mathcal{F}.$$

if $a > d$. $$\{\omega : X(\omega) < a\} = \Omega \in \mathcal{F}.$$

$\therefore$ X is $\mathcal{F}$ - meas ✓.

we will want later to write.

$$X_t = \mathbb{E}[X_T \mid \mathcal{F}_t].$$

↳ this will typically depend on ~~whole~~ S_t

and even it may depend on $\{S_s : 0 \le s \le t\}$.

so we see that $\boxed{X_t \text{ is a RV.}}$

(2) We need to check that

$$\text{(*)} \quad \mathbb{E}[\mathbb{1}_A \mathbb{1}_F] = \mathbb{E}[X \mathbb{1}_F] \quad \text{for every } F \in \mathcal{F}.$$

$$= \mathbb{E}[\mathbb{E}[\mathbb{1}_A | \mathcal{F}] \mathbb{1}_F].$$

$$\mathcal{F} = \sigma(\mathbb{1}_B) = \{\phi, B, B^c, \Omega\}.$$

(i) $F = \phi$ in (*). ✓

$\text{LHS} = 0 = \text{RHS}.$ (since $\mathbb{1}_{\{\phi\}} = 0$)

(ii) $F = \Omega$ in (*). $\mathbb{1}_\Omega = 1.$ ✓

$$\text{LHS} \quad \mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A).$$

$$\text{RHS} \quad \mathbb{E}[X] = \mathbb{E}[\cancel{\mathbb{P}(A)\mathbb{1}} \ \mathbb{P}(A|B)\mathbb{1}_B] + \mathbb{E}[\mathbb{P}(A|B^c)\mathbb{1}_{B^c}].$$

$$= \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c) = \mathbb{P}(A).$$

(iii) $\underline{F = B}$:

LHS : $\mathbb{E}[\mathbb{1}_A \cdot \mathbb{1}_B] = \mathbb{E}[\mathbb{1}_{\{A \cap B\}}] = \mathbb{P}(A \cap B)$. ✓.

RHS $\mathbb{E}[X \mathbb{1}_B] = \mathbb{E}\Big[\mathbb{P}(A|B)\mathbb{1}_B\mathbb{1}_B + \cancel{\mathbb{P}(A)\mathbb{1}}\ \underbrace{\mathbb{P}(A|B^c)\mathbb{1}_{B^c}\mathbb{1}_B}_{=0.}\Big]$.

$= \mathbb{P}(A|B)\,\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(A|B)\,\mathbb{P}(B)$

$= \mathbb{P}(A \cap B)$.

(iv). $\underline{F = B^c}$ LHS : $\mathbb{E}[\mathbb{1}_A \mathbb{1}_{B^c}] = \mathbb{P}(A \cap B^c)$. ✓.

RHS : $\mathbb{E}[X \mathbb{1}_{B^c}] = \mathbb{E}[\mathbb{P}(A|B^c)\mathbb{1}_{B^c}]$

$= \mathbb{P}(A \cap B^c)$

$$\therefore \quad X = \mathbb{E}[\mathbb{1}_A \mid \sigma(\mathbb{1}_B)].$$

$$X = \underbrace{\mathbb{P}(A|B)}_{\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}} \mathbb{1}_B + \mathbb{P}(A|B^c)\,\mathbb{1}_{B^c}.$$

Even in this case where $\mathbb{1}_A, \mathbb{1}_B$ each take on only 2 values this is tedious.

We want rules to compute these!

X, Y are RV's, $\alpha \in \mathbb{R}$.

1) Linearity

$$\mathbb{E}[X + \alpha Y | \mathcal{F}] = \mathbb{E}[X|\mathcal{F}] + \alpha \cdot \mathbb{E}[Y|\mathcal{F}].$$

positivity If $X \leq Y$

$$\mathbb{E}[X|\mathcal{F}] \leq \mathbb{E}[Y|\mathcal{F}].$$

2. ~~$\mathbb{E}$~~ If X is $\mathcal{F}$-meas then

$$\mathbb{E}[X|\mathcal{F}] = X.$$

if X is independent of $\mathcal{F}$, then

$$\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X].$$

(3) (TOWK) If X is $\mathcal{F}$-meas and Y is <u>any</u> RV.

then $\mathbb{E}[XY|\mathcal{F}] = X\,\mathbb{E}[Y|\mathcal{F}]$.

(4) (Tower). $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$ σ-algebras.

$$\mathbb{E}[\,\mathbb{E}[X|\mathcal{F}]\,|\,\mathcal{E}] = \mathbb{E}[X|\mathcal{E}].$$

(5). <u>Indep Lemma.</u> X, Y RV.

X is indep of $\mathcal{F}$ $(X \perp\!\!\!\perp \mathcal{F})$

Y is $\mathcal{F}$-meas.

$f: \mathbb{R}^2 \to \mathbb{R}$.

$$\mathbb{E}[f(X,Y)|\mathcal{F}] = \mathbb{E}^{X}[f(x,Y)] = g(Y).$$

If X has density p_X then..

integrating over.

RV Y.

$$\mathbb{E}[f(X,Y)\mid\mathcal{F}] = \int_{-\infty}^{\infty} f(x,Y)\, p_X(x)\,dx$$

EX: X, Y are Independent RV's.

$X \sim \exp(\lambda)$, $Y \sim \text{unif}\,[0,\lambda]$.

$$Z = f(X,Y) = e^{-XY^2}$$

Find $\mathbb{E}[Z \mid \sigma(Y)] =: \mathbb{E}[Z\mid Y]$.

Sol'n. $$\mathbb{E}[e^{-XY^2}\mid Y] = \int_{-\infty}^{\infty} e^{-xY^2}\, p_X(x)\,dx$$

$$= \int_0^{\infty} \cancel{\lambda e^{-x\lambda}}\, e^{-xY^2}\,\lambda e^{-\lambda x}\,dx.$$

$$= \int_0^\infty \lambda e^{-X(\lambda+Y^2)}\,dX. = \frac{-\lambda}{\lambda+Y^2} e^{-X(\lambda+Y^2)}\Big|_{X=0}^{X=\infty}$$

$$= \frac{\lambda}{\lambda+Y^2} = g(Y) \text{ is a RV.}$$

(6) Law of total Expectation:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}]].$$

Example of two Normal's. on wikipedia where you draw first one value and use that the mean of the 2nd ~~value~~ Normal.

clearly they are dependent.

But true independence from data can be hard to determine.

Lets compute $\mathbb{E}[Z]$.

$$\mathbb{E}[\mathbb{E}[Z|Y]] = \mathbb{E}\left[\frac{\lambda}{\lambda+Y^2}\right] = \int_0^\lambda \frac{\cancel{\lambda}\, 1}{\lambda+y^2}\, dy.$$

$$= \left.\frac{\arctan\left(\frac{Y}{\sqrt{\lambda}}\right)}{\sqrt{\lambda}}\right|_{Y=0}^{Y=\lambda} = \frac{\arctan\sqrt{\lambda}}{\sqrt{\lambda}}.$$

<u>BM</u> : $\{W_t\}_{t \geq 0}$ satisfies.

(i) $W_0 = 0$.

(ii) $W_t \sim N(0,t)$

(iii) $s < t < u < v$ then $W_t - W_s \underset{\uparrow}{\perp\!\!\!\perp} W_v - W_u$.

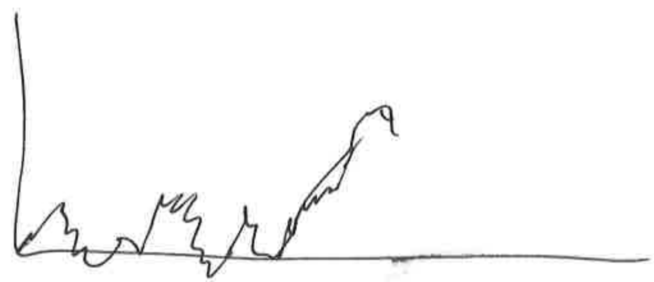
independent

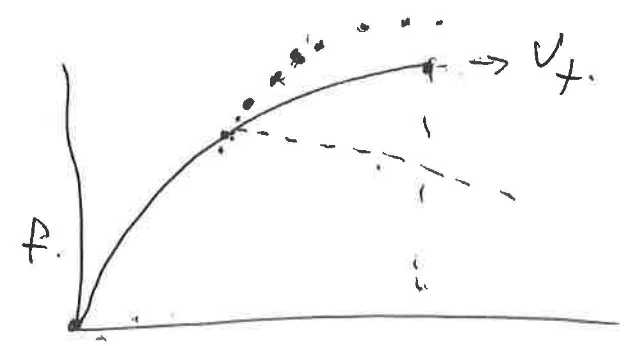
(iv). the function $t \mapsto W_t$ is continuous.

i.e. BM has continuous paths.

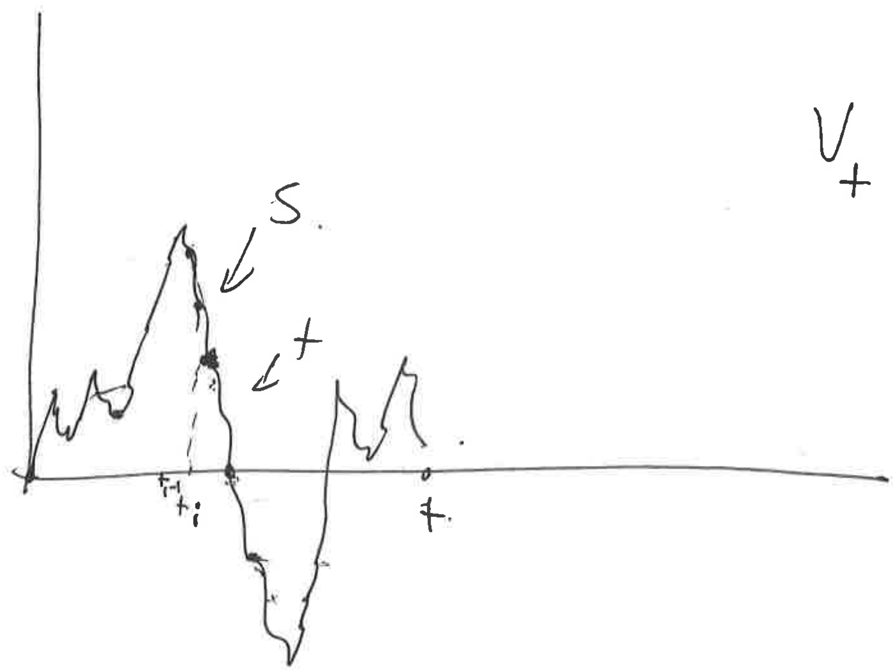
(i) – (iv) define a (standard) Brownian motion.

BM has really "rough" or oscillatory paths.





BM.



$$V_t = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| .$$

$$0 = t_0 < t_1 < \dots < t_n = t$$

$\pi \to$ partition. of $[0,t]$

$$|\pi| = \max_j |t_j - t_{j-1}| .$$

I send $|\pi| \to 0$.

FACT: $V_t = \infty$ for BM.

Warning: $t > s$: $W_t - W_s \perp\!\!\!\perp W_s$.

but $W_s \not\!\perp\!\!\!\perp W_t$

$$\text{so} \quad \mathbb{E}[W_t - W_s \mid \mathcal{F}_s] \overset{\perp\!\!\!\perp}{=} \mathbb{E}[W_t - W_s] = 0.$$

$$\text{but} \quad \mathbb{E}[W_t \mid \mathcal{F}_s] = W_s$$