Last time: Conditional Exp.

$(\Omega, \mathcal{G}, P)$

$\mathcal{F} \subseteq \mathcal{G}$ a σ- sub alg of $\mathcal{G}$.

$X \longrightarrow \mathcal{G}$ meas RV.

$E(X \mid \mathcal{F}) \longrightarrow$ "best approx of X as an $\mathcal{F}$ meas RV"

$$E(X \mid \mathcal{F}) = \begin{cases} ① \text{ Is an } \mathcal{F}\text{- meas RV.} \\ ② \text{ for every } A \in \mathcal{F}, \ \int_A X \, dP = \int_A E(X \mid \mathcal{F}) \, dP \end{cases}$$

Amongst all $\mathcal{F}$-meas RV's, Y, $E(X-Y)^2$ is minimized when $Y = E(X \mid \mathcal{F})$.

Prop: ① Suppose X is an $\mathcal{F}$-meas RV,

then $E(X|\mathcal{F}) = \cancel{EX}$ X

② Suppose X is a $\mathcal{G}$ meas RV that is Independent of $\mathcal{F}$.

(i.e. If $A \in \sigma(X)$, $B \in \mathcal{F}$, then A ind of B
($P(A \cap B) = P(A)P(B)$)).

then $E(X|\mathcal{F}) = EX$

Intuition ②: Say $X = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, $A_i \in \mathcal{G}$

& each A_i is ind of every evnt in $\mathcal{F}$.

Let $B \in \mathcal{F}$ be some event.

Know A_i is ind of B for all i.

Compute $\int_B X \, dP = \int_B \left(\sum a_i \mathbb{1}_{A_i} \right) dP$

$$= \sum a_i \int_B \mathbb{1}_{A_i} \, dP$$

$$= \sum a_i P(A_i \cap B)$$

$$= \sum a_i P(A_i) P(B) \qquad (\because A_i \text{ is ind of } B)$$

$$= P(B) \left[\sum a_i P(A_i) \right] = P(B) \, EX = \int_B (EX) \, dP$$

$$\Rightarrow \int_B X \, dP = \int_B (EX) \, dP \quad \text{for EVERY } B \in \mathcal{G}.$$

$$\left.\begin{array}{c} \| \text{ (def of cond exp)} \\ \int_B E(X \mid \mathcal{G}) \, dP \end{array}\right\} \Rightarrow E(X \mid \mathcal{G}) = EX.$$

(Note: EX is a const ~~rav~~ R.V. & is $\mathcal{G}$ - measurable).

$$\boxed{Z(\omega) = \cancel{f(\omega)} \; 1 \leftarrow \text{const R.V.}}$$

$$\{Z \le \alpha\} \begin{cases} \to \Omega & \alpha \ge 1 \\ \to \phi & \alpha < 1 \end{cases}$$

Independence Lemma: (Examples in Recitation tomorrow).

$\mathcal{G} \longrightarrow f(x,y) \longrightarrow$ fn of two variables

$X \longrightarrow$ ~~Fmeas~~. $\mathcal{G}$ - meas R.V. X is <u>ind</u> of $\mathcal{F}$.

$Y \longrightarrow \mathcal{F}$ meas R.V.

$$E\Big(f(X,Y)\,\Big|\,\mathcal{F}\Big) = g(Y) \quad \text{where}$$

$g = g(y)$ is a (non-random) function defined by

$$g(y) = E f(X, y)$$

Martingales: $\longrightarrow$ Fair Game.

$\hookrightarrow$ ① Filtration:

W = Brownian Motion.

$\sigma(W(t))$. $\longleftarrow$ info obtained by observing $W(t)$

Let $\mathcal{F}_t = \sigma\left(\bigcup_{s \leq t} \sigma(W_s) \right)$.

$= \sigma$-alg generated by $\{ \bigcup_{s \leq t} \sigma(W_s) \}$.

Note: If $s \leq t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$

1. A filtration $\{\mathcal{F}_t\}$ is a collection of σ-algebras such that if $s \le t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

2. The filtration generated by a process X is denoted by $\mathcal{F}_t^X = \sigma\left(\bigcup_{s \le t} \sigma(X_s)\right)$.

3. Most of the time $\mathcal{F}_t$ = filtration generated by B.M.

Def: ① Let $\{\mathcal{F}_t\}$ be a filtration

We say a process X is adapted to the filtration $\{\mathcal{F}_t\}$ if for every $t \geq 0$,

$$X(t) \text{ is an } \mathcal{F}_t\text{-meas R.V.}$$

② We say M is a martingale (wrt the filtration $\{\mathcal{F}_t\}$) if

ⓐ M is adapted (i.e. $M(t)$ is $\mathcal{F}_t$ meas).

& ⓑ If $s \leq t$, $E(M(t) \mid \mathcal{F}_s) = M(s)$

Sub Martingale : ~~we~~ Require $E\left(M(t) \mid \mathcal{F}_s\right) \geq M(s)$.

Super-Martingale : Require $E\left(M(t) \mid \mathcal{F}_s\right) \leq M(s)$.

Prop: If M is a mg, $EM(t) = EM(0)$

Proof: Know $M(0) = E\left(M(t) \mid \mathcal{F}_0\right)$ (mg prop).

$$\Rightarrow EM(0) = E\,E\left(M(t) \mid \mathcal{F}_0\right) = EM(t).$$

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Warning: If $EM(t) = EM(0)$ for every t must M be a mg? NO (Eg in Recitation)

Eg: B.M. is a mg

Filtration $\longrightarrow$ Brownian filtration.

$W \longrightarrow$ std Brownian motion.

To check BM is a mg, need to check $(*)$.

for every $s \leq t$, $\boxed{E(W(t) \mid \mathcal{F}_s) = W(s)}$.

Proof of $(*)$: $E(W(t) \mid \mathcal{F}_s) = E(W(t) - W(s) + W(s) \mid \mathcal{F}_s)$

$$= E(W(t) - W(s) \mid \mathcal{F}_s) + E(W(s) \mid \mathcal{F}_s).$$

$$E(W(t) - W(s)) \quad + \quad W(s)$$

(∵ ① $W(t) - W(s)$ is independ of $\mathcal{F}_s$.
(ind inc)

& ② $W(s)$ is $\mathcal{F}_s$ meas).

$$\Rightarrow E(W(t) \mid \mathcal{F}_s) = W(s) + 0 = W(s)$$
//.

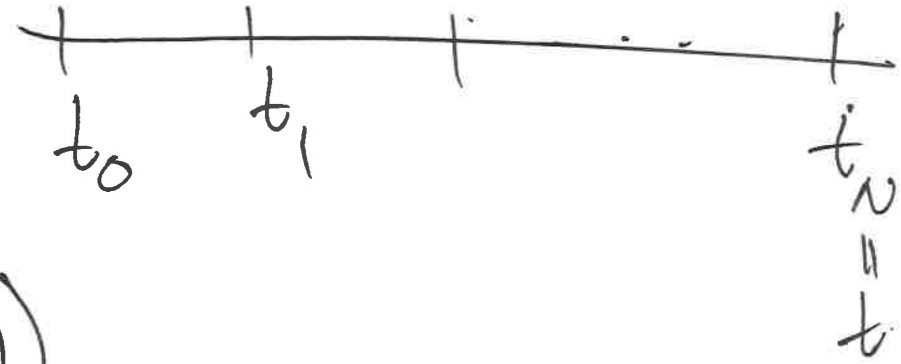
Stochastic Integration:

Stock $\longrightarrow$ price at time $t = S(t)$.

$\Delta(t)$ = position on the stock = # shares held at time t.

Only trade at times $0 = t_0 < t_1 < \cdots < t_N$.

Change in wealth



$$= X(t_N) - X(0)$$

$$= \sum_{i=0}^{n-1} \Delta(t_i)\left(S(t_{i+1}) - S(t_i)\right)$$

$$P = \{ 0 = t_0 < t_1 < t_2 \cdots t_N = t \}$$

partition of $[0, t]$.

$$\|P\| = \max_i \; t_{i+1} - t_i$$

Trade ctsly:

Expect $X(t) - X(0) = \lim_{\|P\| \to 0} \sum \Delta_i(t) \left(S(t_{i+1}) - S(t_i) \right)$.

$$= \int_0^t \Delta(s)\, dS(s).$$

(Riemann-Stiltjes Integral).

From <u>calculus</u>: This limitg process only works for "processes of finite 1st variation".

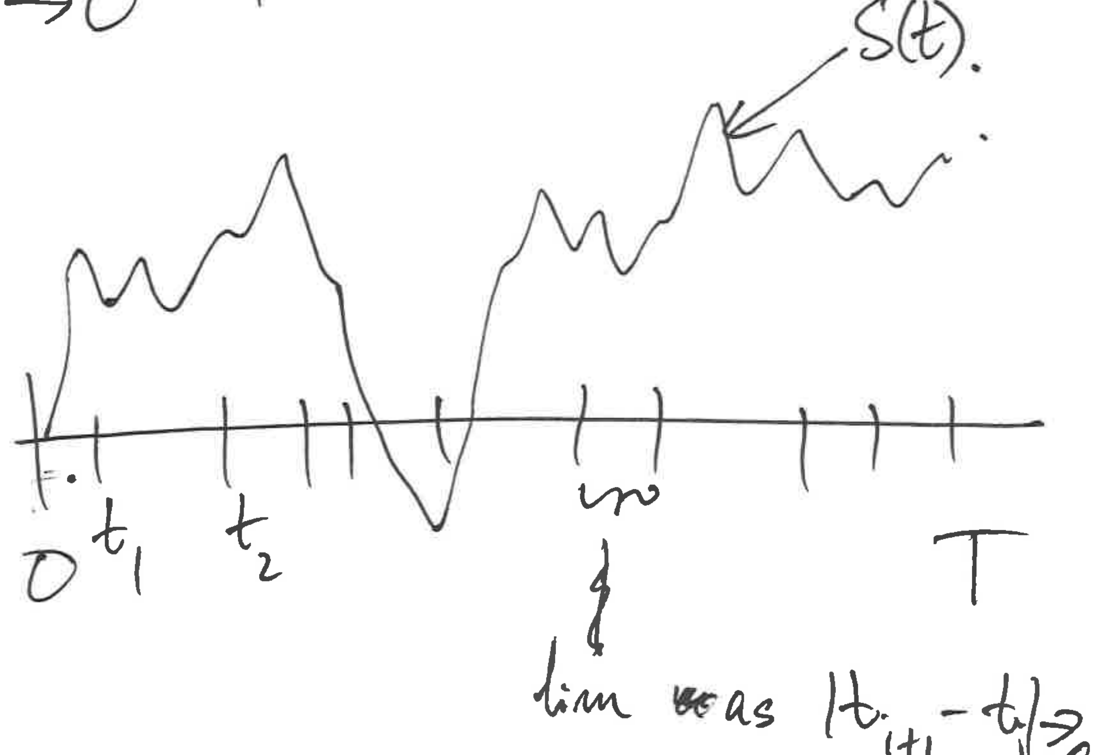
Stock prices typically dont have this.

Instead: Use the "Itô Integral".

① First variation.

$S \to$ some process.

$$V_{[0,T]}(S) \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} |S(t_{i+1}) - S(t_i)|.$$



Claim: Brownian Motion does NOT have finite 1^{st} variation.

Prop:
$$\lim_{n\to\infty} \sum \left| W\left(\frac{(i+1)T}{n}\right) - W\left(\frac{iT}{n}\right) \right| = +\infty$$
almost surely!

(Proof in notes).
No time in lec.

What Saves Us: Quadratic Variation

Def: Let M be any process.

Define the quadratic variation of M by

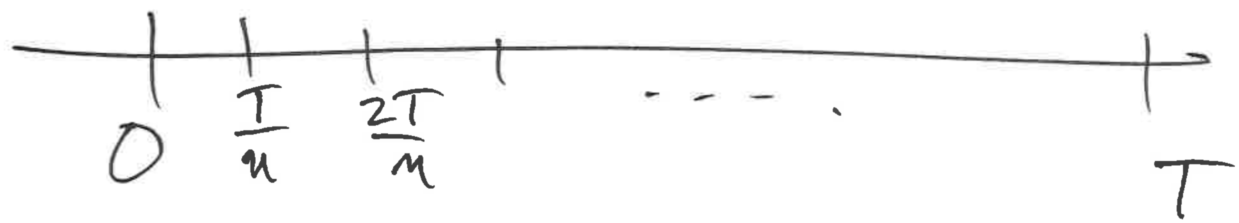
$$[M, M](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left(M(t_{i+1}) - M(t_i) \right)^2 .$$

Compute $[W, W]$:

Let W = std 1D B.M.

Claim: $[W, W](T) = T$

Proof: For simplicity assume $t_i = \frac{T \cdot i}{n}$ (uniform partition).



Let $\Delta_i W = W(t_{i+1}) - W(t_i)$.

$$= W\left(\frac{T(i+1)}{n}\right) - W\left(\frac{T i}{n}\right).$$

NTS: $\lim\limits_{n \to \infty} \sum (\Delta_i W)^2 = T.$

Note:

$$\sum_{i=0}^{n-1} (\Delta_i W)^2 - T = \sum_{i=0}^{n-1} \underbrace{\left[\left(W\left(\frac{(i+1)T}{n}\right) - W\left(\frac{iT}{n}\right)\right)^2 - \frac{T}{n}\right]}_{\xi_i}$$

Note: ξ_i are all iid RV's.

$$\xi_i \sim \left[N\left(0, \frac{T}{n}\right)^2 - \frac{T}{n}\right].$$

$$\Rightarrow E\xi_i = 0 \ \& \ E\xi_i^2 = \frac{T^2}{n^2}\left(E N(0,1)^4 - 1\right).$$

$$\Rightarrow \operatorname{Var}\left(\sum_{i=0}^{n-1} \xi_i\right) = \sum_{i=0}^{n-1} \operatorname{Var}(\xi_i).$$

$$= \sum_{i=0}^{n-1} \frac{T^2}{n^2} \left(E\, N(0,1)^4 - 1 \right)$$

$$\xrightarrow{n \to \infty} 0$$

$$\Rightarrow \lim_{n \to \infty} \operatorname{Var} \left(\sum_{i=0}^{n-1} (\Delta_i W)^2 - T \right) \longrightarrow 0$$

$$\Rightarrow \underbrace{\lim_{n \to \infty} \sum_{i=0}^{n-1} (\Delta_i W)^2}_{[W,W](T)} = T$$

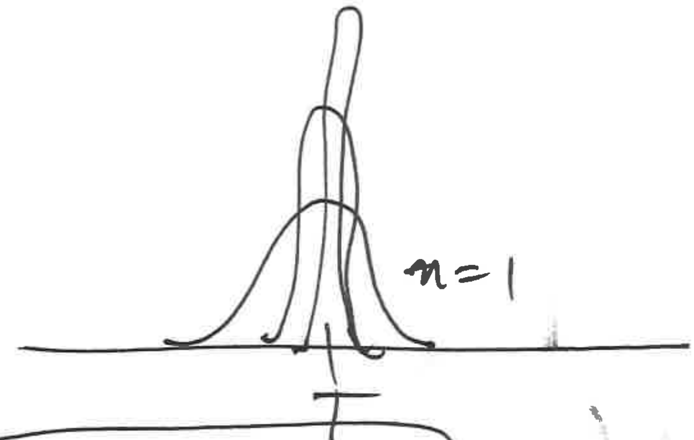
//.

$$E N(0, \tau^2) = 0$$

$$\lim_{n\to\infty} \mathrm{Var}(X_n - T) = 0$$

$$E N(0, \tau^2)^2 = \tau^2,$$

$$= \mathrm{Var}(N(0, \tau^2)),$$



$$\xi_i \sim \left[N\left(0, \frac{T}{n}\right)^2 - \frac{T}{n}\right].$$

$$E\,\xi_i^2 = E\left(N\left(0, \frac{T}{n}\right)^4 + \frac{T^2}{n^2} - 2\frac{T}{n} N\left(0, \frac{T}{n}\right)^2\right),$$

$$= E N\left(0, \frac{T}{n}\right)^4 + \cancel{\frac{T^2}{n^2}} - \cancel{2}\frac{T^2}{n^2}$$