①

$\Omega = [0, 1)$

$\underline{\mathbb{P}[X_n(y) \leq a]} =$ $\qquad \forall a \in \mathbb{R}.$

- X, Y are not (linearly) correlated. provid $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 0$
- X, Y are independent provided

$$\mathbb{P}[X \leq a, Y \leq b] = \mathbb{P}[X \leq a]\,\mathbb{P}[Y \leq b]$$

- X, Y are independent. $\Rightarrow$ X, Y are uncorrelated
- ~~X, Y are uncorrelated $\Rightarrow$ X, Y are independent~~

X, Y are uncorrelated, $(X, Y) \sim N(\mu, \Sigma)$

$$\Sigma \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$\Rightarrow$ X, Y are independent.

Prob 1. $X \sim N(0,1)$, Y, $\mathbb{P}[Y=1] = \frac{2}{3}$, $\mathbb{P}[Y=-1] = \frac{1}{3}$

$Z = XY$, $X \perp\!\!\!\perp Y$

1) What's cdf Z?

2) $Z \perp\!\!\!\perp Y$? $Z \perp\!\!\!\perp X$?

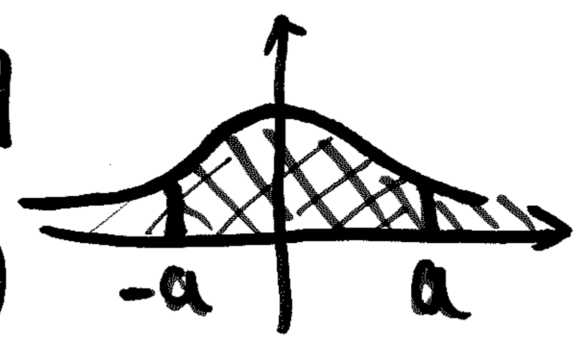
Proof: 1) $\forall a \in \mathbb{R}$

$$\mathbb{P}[Z \le a] = \mathbb{P}[Y=1]\,\mathbb{P}[X \le a] + \mathbb{P}[Y=-1]\,\mathbb{P}[-X \le a]$$

$$= \frac{2}{3}\mathbb{P}[X \le a] + \frac{1}{3}\mathbb{P}[X \ge -a]$$

Since $\mathbb{P}[X \le a] = \mathbb{P}[X \ge -a]$

$\Rightarrow \mathbb{P}[Z \le a] = \mathbb{P}[X \le a]$, $Z \sim N(0,1)$



$$2)\ \mathbb{P}[Z \le a,\ Y=1] = \mathbb{P}[X \le a,\ Y=1] \qquad \forall a \in \mathbb{R}$$

④

$$= \mathbb{P}[Y=1]\ \mathbb{P}[X \le a]$$

$$= \mathbb{P}[Y=1]\ \mathbb{P}[Z \le a]$$

$Z \perp\!\!\!\perp Y.$

on $\{Y=1\}$ $\quad Z = XY = X \cdot 1 = X$

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3) $\cancel{\mathbb{E}[Z \mid X=1]}$ $\quad$ ~~$\forall a \in \mathbb{R},\ b \in \mathbb{R}$~~, $\cancel{a}\ 1 \le b \le 0$

$\mathbb{P}[Z \le a,\ X \le b] =$

on $\{X \le \cancel{b}\ 0\}$, if $Y=1$, $\quad Z \le \cancel{b}\ 0$

~~if $Y=-1$, $\quad Z \ge \cancel{b}\ 0 > 0$~~

$\mathbb{P}[Z \le \cancel{a}\ 1,\ X \le \cancel{b}\ 0] = \mathbb{P}[Y=1,\ X \le -1] = \frac{2}{3} \cdot \mathbb{P}[X \le -1]$

$\mathbb{P}[X \le 0] = \frac{1}{2}$, $\mathbb{P}[X = 0] = 0$ ⑤

$\mathbb{P}[Z \le -1] = \mathbb{P}[X \le -1]$

$\mathbb{P}[Z \le -1, X \le 0] = \frac{2}{3} \cdot \mathbb{P}[X \le -1]$

$\neq \frac{1}{2} \mathbb{P}[X \le -1]$

Q? if $\mathbb{P}[Y = 1] = \mathbb{P}[Y = -1] = \frac{1}{2}$

does $X \perp\!\!\!\perp Z$ hold?

In stat Gaussian mixture model.

3). $\mathbb{E}[Z \mid Y=1]$, $\mathbb{E}[Z \mid X]$

- $Z \perp\!\!\!\perp Y$ $\mathbb{E}[Z \mid Y=1] = \mathbb{E}[Z] = 0$.

- $\mathbb{E}[Z \mid X]$

$\mathbb{E}[Z \mid X, Y=1] = X$

if I know X, $Y=1 \Rightarrow Z = X \cdot Y = X$

$\mathbb{E}[Z \mid X, Y=-1] = -X$

$\mathbb{E}[Z \mid X] = \mathbb{P}[Y=1]\,\mathbb{E}[Z \mid X, Y=1] + \mathbb{P}[Y=-1]\,\mathbb{E}[Z \mid X, Y=-1]$

$= \frac{1}{3} X$

4) $\mathbb{E}[Y|Z]$, $\mathbb{E}[X|Z]$ ⑦

$Y \perp\!\!\!\perp Z$, $\mathbb{E}[Y|Z] = \mathbb{E}[Y] = \frac{2}{3}\cdot 1 + \frac{1}{3}(-1) = \frac{1}{3}$

$\mathbb{E}[X|Z, Y=1] = Z$

$\mathbb{E}[X|Z, Y=-1] = -Z$

$\mathbb{E}[X|Z] = \frac{1}{3}Z$

$\mathbb{E}[X|Y]$.

$\forall Z$ is measurable $\sigma(Y)$

NTS $\mathbb{E}[(X - \mathbb{E}[X|Y])^2] \leq \mathbb{E}[(X-Z)^2]$

$a^2+b^2=c^2$

M'gale $\{M_n\}_{n\geq 1}$ $\qquad \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$ a.s

$\{M_t\}_{t\geq 0}$. $\quad \forall\, 0 \leq s \leq t$

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{a.s}$$

Markov $\{M_n\}_{n\geq 1}$, $\quad \mathbb{P}[X_{n+1} \leq a \mid X_0, \cdots, X_n]$

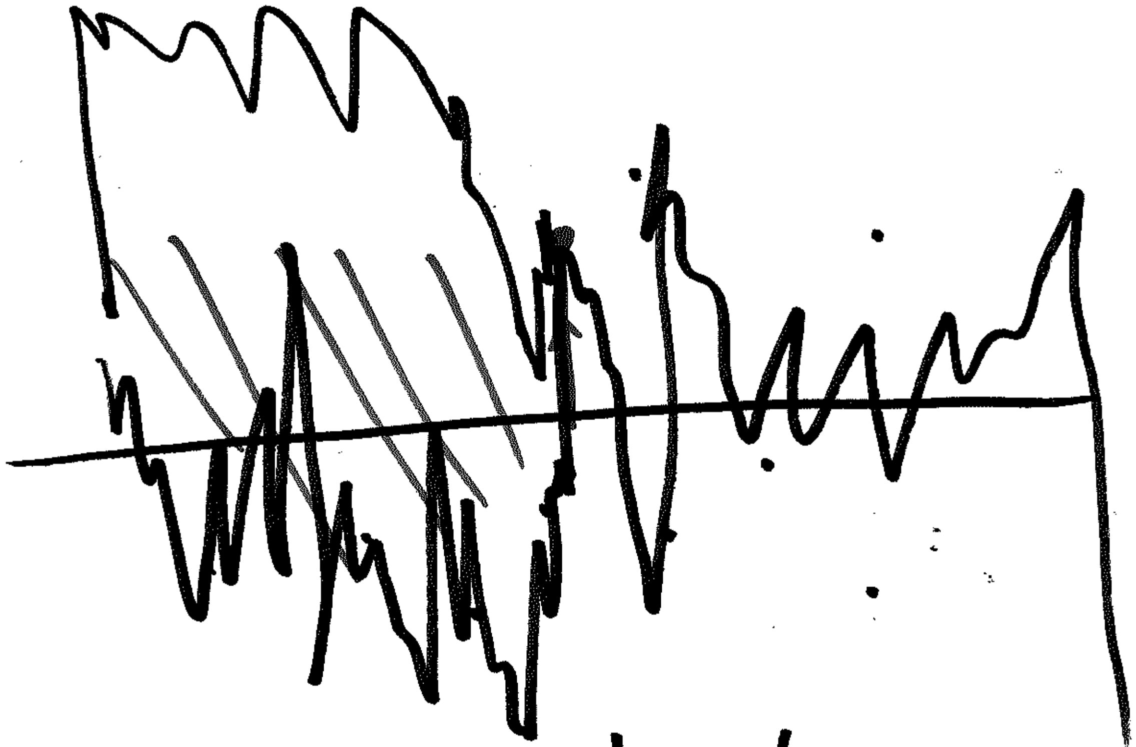
$$= \mathbb{P}[X_{n+1} \leq a \mid X_n]$$

information at time s

$\{M_t\}_{t\geq 0}$, $\quad \forall\, 0 \leq s \leq t$

$$\mathbb{P}[X_t \leq a \mid \mathcal{F}_s] = \mathbb{P}[X_t \leq a \mid X_s]$$

information collected until time s



given now, what's happening in the future

⫫ what happened in the past