New Recitation Time: 12:30 — 2:00 pm Friday

Independence: B.M. $\to$

① Cts process.

$\boxed{\text{② Independent increments.}}$

③ $W(t) - W(s) \sim N(0, \sigma^2(t-s))$.

Std B.M. $\sigma = 1$. $(W(t) - W(s) \sim N(0, t-s))$.

Q: Compute $E\, W(s) W(t)$ when $s \le t$

Ans: $\underline{\quad s \quad}$

$$E\, W(s)W(t) = E\, W(s)\big((W(t)-W(s)) + W(s)\big).$$

$$= E\, W(s)^2 + E\, W(s)\,(W(t)-W(s))$$

$$= s + E\,W(s)\; E(W(t)-W(s)). \qquad (\because W(s) \text{ ind of } W(t)-W(s))$$

$$= s$$

In general: $E\, W(s) W(t) = s \wedge t$ (s minimum t).

Today: Conditional Expectation.

$\rightarrow$ Risk Neutral pricing formula.

Derivative security with payoff $V(T)$ at time T.

Price at time $t < T$ is given by

$$\rightarrow \tilde{E}\left(V(T) \cdot D(T-t) \;\middle|\; \mathcal{F}_t \right)$$

new measure.

discount factor

Conditional Expectation.

σ-alg containing events that are known up to time t.

Conditional Expectation: $(\Omega, \mathcal{G}, P)$

$\mathcal{G} \to \sigma$-alg. (events whose prob is known).

Let $\mathcal{F} \subseteq \mathcal{G}$ be a σ-sub alg of $\mathcal{G}$.

1. $\mathcal{F} \subseteq \mathcal{G}$ (i.e. $A \in \mathcal{F} \Rightarrow A \in \mathcal{G}$).
2. $\mathcal{F}$ is a σ-alg.

Eg: Say X is a $\mathcal{G}$-meas R.V.

Let $\mathcal{F} = \sigma(X) \leftarrow \sigma$-alg generated by X.

Clearly $\sigma(X) \subseteq \mathcal{G}$.

But $\sigma(X) \neq \mathcal{G}$ in most cases.

$\mathcal{G} \longrightarrow$ "all infortion"

$\mathcal{F} \longrightarrow$ "information available to me".

Say Z is a $\mathcal{G}$-meas R.V.

$E(Z \mid \mathcal{F})$ = conditional expectation of the R.V. Z given the σ-alg $\mathcal{F}$.

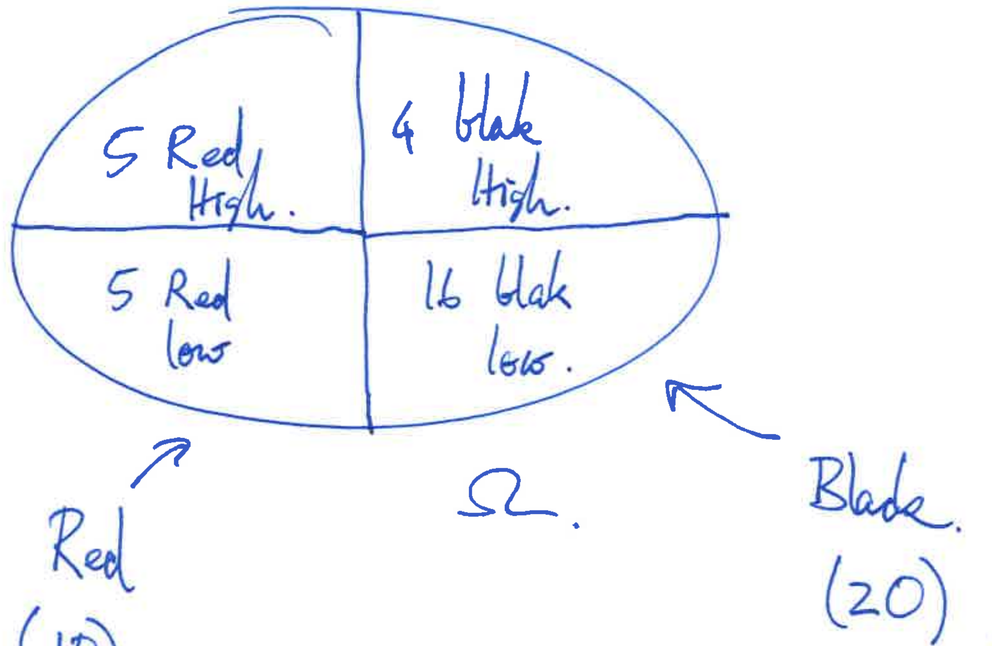
= The "best approximation" of the R.V. Z by an $\mathcal{F}$ meas R.V.

= The (unique) $\mathcal{F}$-meas. R.V. X which minimizes $E(Z-Y)^2$ over all $\mathcal{F}$-meas RV's Y.

Eg: Ω = 30 cards.

$$\left(\Omega,\ \overset{\mathcal{G}}{\overbrace{\mathcal{P}(\Omega)}},\ P\right)$$

$\mathcal{P}(\Omega)$ ↑ all subsets of Ω



<u>Game:</u>

→ +1 \$ High card. (10)

→ −1 \$ low card.

$$X = \mathbb{1}_H - \mathbb{1}_L$$

H = {High cards}

L = {low cards}.

Infortion: Whether card drawn is Red or black.
(not told High or low).

$$\mathcal{F} = \{\phi, \Omega, R, B\} \qquad R = \{\text{~~one~~ red cards}\}$$
$$B = \{\text{black cards}\}.$$

Guess: Best approx of X as an $\mathcal{F}$ meas R.V.

is $\underset{\cancel{1/2}}{0}\,\mathbb{1}_R + \underset{1/5}{\frac{4-16}{20}}\,\mathbb{1}_B$

Guess $E(X \mid \mathcal{F}) = 0\,\mathbb{1}_R - \frac{3}{5}\,\mathbb{1}_B$.

You check: $\min E\Big(X - (\underbrace{\alpha \mathbb{1}_R + \beta \mathbb{1}_B}_{\text{all } \mathcal{F} \text{ meas R-V's}})\Big)^2$ over all α & β

I check: $\alpha = 0$
$\beta = -3/5$.

<u>Claim:</u> For every $A \in \mathcal{G}$.

$$\underbrace{\int_A X \, dP}_{E(\mathbb{1}_A X)} = \underbrace{\int_A E(X|\mathcal{G}) \, dP}_{E(\mathbb{1}_A E(X|\mathcal{G}))}.$$

Note: $\cancel{\int_B}$ Check for $A = B$ (Black cards)

$$\int_B X \, dP = E(\mathbb{1}_B X) = \cancel{\tfrac{3}{8}} = E(\cancel{4}\,\mathbb{1}_{HB} - \mathbb{1}_{LB}) = \frac{4}{30} - \frac{16}{30} = \frac{-12}{30} = \frac{-4}{10}$$

$$\int_B E(X|\mathcal{G}) \, dP = E\left(\mathbb{1}_B \left(0 \mathbb{1}_R - \frac{3}{5} \mathbb{1}_B\right)\right) = -\frac{3}{5} P(B) = \cancel{\tfrac{18}{10}} = -\frac{2}{5}$$

More useful characterization of cond exp:

$X \longrightarrow \mathcal{G}$ meas RV.

$\mathcal{F} \subseteq \mathcal{G}$ a σ- sub alg.

$E(X \mid \mathcal{F})$ is the (unique) $\boxed{\underline{\mathcal{F}} \text{ meas R.V.}}$

such that for every $A \in \mathcal{F}$ we have

$$E(\mathbb{1}_A X) = E(\mathbb{1}_A E(X \mid \mathcal{F}))$$

$$\boxed{\Leftrightarrow \quad \int_A X \, dP = \int_A E(X \mid \mathcal{F}) \, dP.}$$

partial averaging property.

Q1: Is $EX = E\,E(X|\mathcal{G})$

Remark $E(X|\mathcal{G})$ is a R.V.

True:

Proof: Q: $\Omega \in \mathcal{G}$? ✓

$\Rightarrow$ Partial av works with $A = \Omega$.

$$\Rightarrow \underbrace{\int_\Omega X\, dP}_{EX} = \underbrace{\int_\Omega E(X|\mathcal{G})\, dP}_{E\left(E(X|\mathcal{G})\right)}.$$

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Eg: If $A_1, A_2 \cdots A_n$ are disj & $\Omega = \cup A_i$.

& $\mathcal{G} = \sigma(A_1, A_2, \ldots A_n) = \{\phi, A_i, A_i \cup A_j, A_i \cup A_j \cup A_k, \text{etc}\}$.

Then $E(X \mid \mathcal{G}) = \sum_{i=1}^{n} 1\!\!1_{A_i} \Big(\frac{1}{P(A_i)} \underbrace{\int_{A_i} X \, dP}_{\text{number}} \Big)$. (Bayes Rule)

Properties of Conditional Exp:

1. If X, Y are 2 RV's. ($\mathcal{G}$-meas) & $\alpha \in \mathbb{R}$.

   then $E(X + \alpha Y \mid \mathcal{F}) = E(X \mid \mathcal{F}) + \alpha E(Y \mid \mathcal{F})$.

2. (Positivity) If $X \leq Y$, then $E(X \mid \mathcal{F}) \leq E(Y \mid \mathcal{F})$.

3. If X is $\mathcal{F}$ meas, & Y is $\mathcal{G}$ meas.

   then $E(XY \mid \mathcal{F}) = X E(Y \mid \mathcal{F})$.

4. (Tower Property). Say $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$ are 3 σ alg.

   then $E(X \mid \mathcal{E}) = E(E(X \mid \mathcal{F}) \mid \mathcal{E})$.

Proposition: X be a $\mathcal{G}$ meas R.V. (least square approx).
$\mathcal{F} \subseteq \mathcal{G}$ a σ-sub alg.

$E(X \mid \mathcal{F})$ = cond exp of X given $\mathcal{F}$.
(i.e. an $\mathcal{F}$ meas R.V. such that $\int_A X \, dP = \int_A E(X \mid \mathcal{F}) \, dP$
for every $A \in \mathcal{F}$).

Then amongst all $\mathcal{F}$ meas R.V.'s Y,

$E(X-Y)^2$ is minimized when $Y = E(X \mid \mathcal{F})$.

$$\| $$

$$E\left[(X-Y)^2\right].$$

Proof: Let Z be any $\mathcal{G}$ meas R.V..

Claim: $E(X-Z)^2 \geq E(X - E(X|\mathcal{G}))^2$.

(Claim $\Rightarrow$ Prop since $E(X|\mathcal{G})$ is $\mathcal{G}$_meas.)

$$E(X-Z)^2 = E(X - E(X|\mathcal{G}) + E(X|\mathcal{G}) - Z)^2.$$

$$= E(X - E(X|\mathcal{G}))^2 + E(\cancel{x}\, E(X|\mathcal{G}) - Z)^2 \quad +\text{ve}.$$

$$+\ 2\,\underbrace{E(X - E(X|\mathcal{G}))(E(X|\mathcal{G}) - Z)}_{\text{Claim } = 0}$$

~~Claim~~:

$$(\Rightarrow \text{propostion}).$$

Note $E\big(X-E(X|\mathcal{G})\big)\big(E(X|\mathcal{G})-Z\big)$

$$= E\,E\Big(\big(X-E(X|\mathcal{G})\big)\big(E(X|\mathcal{G})-Z\big)\,\Big|\,\mathcal{G}\Big). \qquad [\because Q1].$$

$\mathcal{G}$ meas.

$$\overset{(\text{Prop } 3)}{=} E\big(E(X|\mathcal{G})-Z\big)\,E\big(X-E(X|\mathcal{G})\,\big|\,\mathcal{G}\big).$$

$$= E\big(E(X|\mathcal{G})-Z\big)\Big[E(X|\mathcal{G})-\underbrace{E\big(E(X|\mathcal{G})\,|\,\mathcal{G}\big)}_{E(X|\mathcal{G})}\Big].$$

$$= 0 \quad //.$$