David Itkin.

9:30 - 11:00 AM Every Friday.

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OH : Pitt: Wednesday 10-11 AM in Office.

NY : Wednesday 11-12 on Canvas.

↳ Let me know atleast 1 hour in advance.

Main goal: To develop continuous-time financial models

Math	Applications.
- Brownian Motion	- Black-Scholes Model.
- Stochastic Integration ⟶ Ito's formula's.	- Risk-neutral pricing
- Girsanov's theorem	

Today: σ-algebras, RV's, Probability measures and Independence.

Recall: σ-algebra $\mathcal{F}$ it's a set of subsets of Ω that satisfies

(i) $\emptyset \in \mathcal{F}$.

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(iii) $\{A_n\}_{n\geq 1}^{\infty} \subseteq \mathcal{F} \Rightarrow \bigcup_{n\geq 1}^{\infty} A_n \in \mathcal{F}$.

T or F $\quad \mathcal{F}, \mathcal{G}$ σ-algebras. Is $\mathcal{F} \cap \mathcal{G}$ a σ-algebra?

(i) $\phi \in \mathcal{F}, \quad \phi \in \mathcal{G} \Rightarrow \phi \in \mathcal{F} \cap \mathcal{G}.$

(ii) $A \in \mathcal{F} \cap \mathcal{G}. \Rightarrow \begin{cases} A^c \in \mathcal{F} \\ A^c \in \mathcal{G}. \end{cases} \Rightarrow A^c \in \mathcal{F} \cap \mathcal{G}.$

(iii) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F} \cap \mathcal{G}. \Rightarrow \begin{cases} \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{G}. \end{cases} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \cap \mathcal{G}.$

T or F. $\mathcal{F}, \mathcal{G}$ σ-algebras. Is $\mathcal{F} \cup \mathcal{G}$ a σ-algebra?

(iii) Can fail

Ex (Coin Toss).

$$\Omega = \{HH, HT, TH, TT\}.$$

$$\mathcal{F} = \{\phi, \Omega, \{HH\}, \{TT, HT, TH\}\} = \sigma(\{HH\}).$$

$$\mathcal{G} = \{\phi, \Omega, \{TT\}, \{HH, HT, TH\}\} = \sigma(\{TT\}).$$

$$\mathcal{F} \cup \mathcal{G} = \{\phi, \Omega, \{HH\}, \{TT\}, \{TT, HT, TH\}, \{HH, HT, TH\}\}.$$

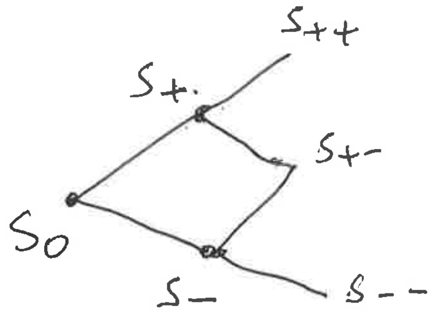
$\{HH\}, \{TT\} \in \mathcal{F} \cap \mathcal{G}$.

but $\{HH, TT\} \notin \mathcal{F} \cap \mathcal{G}$.

In general $\mathcal{F} \cup \mathcal{G}$ is not a σ-algebra and

$\sigma(\mathcal{F} \cup \mathcal{G})$ is bigger that $\mathcal{F} \cup \mathcal{G}$.

EX Binomial Tree – Discret time Finance.



$\Omega = \{$ all possible stock price paths $\}$.

$S_0 \to S_+$ $\quad S_+ \to S_{++}$.

$= \{(S_+, S_{++}), (S_+, S_{+-}), (S_-, S_{+-}), (S_-, S_{--})\}$.

σ-algebras represent information:

Time 0 $\mathcal{F}_0 = \{\phi, \Omega\}$. $\rightsquigarrow$ no information at time 0.

Time 1 $\mathcal{F}_1 = \{\phi, \Omega, \{(S_+, S_{++}), (S_+, S_{+-})\}, \{(S_-, S_{--}), (S_-, S_{+-})\}$

$\hookrightarrow$ If stock price increased.

$\hookrightarrow$ $\{(S_0, S_+, S_{++})\}\ (S_0, S_+, S_{+-})\}$.

$\hookrightarrow$ If stock price decreased

Time 2 $\mathcal{F}_2 = \mathcal{P}(\Omega) = \{$ all subsets of $\Omega\}$.

why does for example $\{(S_+, S_{++})\}$ not appear in $\mathcal{F}_1$?

Because we do not know the time 2 value at time 1.

Random Variables:

Def: $(\Omega, \mathcal{F}, \mathbb{P})$ $\quad X : \Omega \to \mathbb{R}$ (function) is a RV if

$$\forall a \in \mathbb{R} : \{ \omega \in \Omega : X(\omega) < a \} \in \mathcal{F}.$$

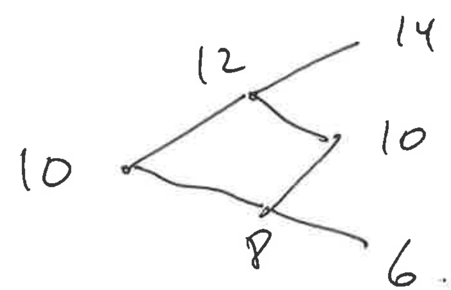
↳ shorthand $X^{-1}(-\infty, a)$.

Ex 1) $\Omega = \{$all outcomes of N coin tosses$\}$.

$\mathcal{F} = \mathcal{P}(\Omega) \quad X = \#$ Heads after N coin tosses

X is a RV!

2 Stock price Example:



$\mathcal{F}_1$ and $\mathcal{F}_2$ are as before.

$$\mathcal{F}_1 = \{\phi, \Omega, \{(12,14),(12,10)\}, \{(8,10),(8,6)\}\}.$$

$$\mathcal{F}_2 = \mathcal{P}(\Omega).$$

stock price value at time 1.

Q1: Is $X(\omega) := \dfrac{S_1(\vec{\omega}) - 10}{10}$ a RV wrt $\mathcal{F}_1$?

Yes   Intuitively: we should be able to determine the ~~rate of return at time 1~~ 1 period rate of return at time 1.

Formally: Check def'n :

$$X_1(\omega) = \begin{cases} \frac{1}{5} & \text{if } S_1 = 12 \\ -\frac{1}{5} & \text{if } S_1 = 8. \end{cases}$$ → takes on 2 values.

Need to examine $X^{-1}(-\infty, a) = \{\omega \in \Omega : X(\omega) < a\}$.
for every $a \in \mathbb{R}$.

First take $a \le -\frac{1}{5}$:

then for such a

$$\{ w \in \Omega : X(\omega) < a \} = \phi \in \mathcal{F}_1$$

take $-\frac{1}{5} < a \le \frac{1}{5}$

$$\{ w \in \Omega : X(\omega) < a \} = \{ (8,6), (8,10) \} \in \mathcal{F}_1$$

$a > \frac{1}{5}$

~~$\{(8,6)\},\{$~~ $\{(8,6), (8,10), (12,10), (12,14)\}$

$$\{ w \in \Omega : X(\omega) < a \} = \Omega \in \mathcal{F}_1$$

$\therefore$ X is a RV.

Q2 : Is $X = \frac{S_1 - 10}{10}$ a RV with respect to $\mathcal{F}_2$?

Yes -- we should be able to determine period 1 rate of return at time 2.

Let's check! For every $a \in \mathbb{R}$

$\{\omega \in \Omega : X(\omega) < a\} \in \mathcal{F}_1 \subseteq \mathcal{F}_2$. ✓

If $\mathcal{F} \subseteq \mathcal{G}$ are σ- algebras and

X is a RV wrt $\mathcal{F}$ then it also is a RV with respect to $\mathcal{G}$.

Q3 $Y = \frac{S_2 - 10}{10}$. Is Y a RV with respect to $\mathcal{F}_1$?

No!

$Y = \frac{2}{5}$ or $Y = 0$ or $Y = -\frac{2}{5}$

take $a \in (-\frac{2}{5}, 0)$

$\{\omega \in \Omega : \cancel{X} Y(\omega) < a\} = \{(8,6)\} \notin \mathcal{F}_1$.

$\therefore$ Y is not a RV with respect to $\mathcal{F}_1$.

Q4: ~~Y~~ Is Y a RV with respect to $\mathcal{F}_2$?

Yes! You can check as an exercise!

This example in discrete time is good for intuition because everything is explicit.

I.e. we can easily write down all the σ- algebras.

In cont. time it is impossible to write down σ- algebras explicitly. . .

we will look at $S_t \in [0,\infty)$ $\forall\, t \in \mathbb{R}^+$.

<u>Ex</u> (Borel σ-algebra). Most important σ-algebra on $\mathbb{R}$, or on $[0,1]$.

$\mathcal{B}$ = smallest σ-algebra containing all intervals of the form $(-\infty, a)$. for every $a \in \mathbb{R}$

Claim: if ~~$a \geq b$~~ $a < b$. then $[a,b] \in \mathcal{B}$.

$\forall n \in \mathbb{N}\ (-\infty, b + \frac{1}{n}) \in \mathcal{B}$.

Aside

Intersections of countably many sets are in σ-algebras as well as unions.

why? $\{A_n\}_{n=1}^{\infty} \in \mathcal{F} \Rightarrow \{A_n^c\}_{n=1}^{\infty} \in \mathcal{F}$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{F} \Rightarrow \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$$

$$\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c = \bigcap_{n=1}^{\infty} A_n$$

of the form $(-\infty, a)$ where $a = b + \frac{1}{n}$.

$$\bigcap_{n=1}^{\infty} \left(-\infty, b+\tfrac{1}{n}\right) \in \mathcal{B}$$
$$\parallel$$
$$(-\infty, b]$$

Also $(-\infty, a) \in \mathcal{B} \Rightarrow (-\infty, a)^c = [a, \infty) \in \mathcal{B}$.

$\therefore$ $[a,b] = [a,\infty) \cap (-\infty, b] \in \mathcal{B}$.

things like. $\displaystyle\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \cdots A_{nmk} \in \mathcal{B}$ if $A_{nmk} \in \mathcal{B}$

The point is $\mathcal{B}$ contains many, many sets and one cannot explicitly write down what it is.

Probability: Given $(\Omega, \mathcal{F})$ you want to assign probabilities to events in $\mathcal{F}$.

Def $\mathbb{P} : \mathcal{F} \to [0,1]$ is a prob. meas if

(i) $\mathbb{P}(\emptyset) = 0$ (or equiv $\mathbb{P}(\Omega) = 1$).

(ii) $\{A_n\}_{n=1}^{\infty}$ are pairwise disjoint then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Back to Tree: pick $p \in [0,1]$ define on $\mathcal{F}_1$

$\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$, $\cancel{\mathbb{P}(\{(10,12)}$, $\mathbb{P}(\{(12,14),(12,10)\}) = p$

$\mathbb{P}(\{(8,6),(8,10)\}) = 1-p.$

Then $Z = \begin{cases} 1 & \text{if } S_1 = 12 \\ 0 & \text{if } S_1 = 8 \end{cases}$ is a Bernoulli RV with prob of success p under $\mathbb{P}$.

Independence $(\Omega, \mathcal{F}, \mathbb{P})$.

Def We say two σ-algebras $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$ are independent if $\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2$ $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

We say RV's X and Y are independent if $\sigma(X)$ is independent of $\sigma(Y)$.

$\sigma(X)$ is the σ-algebra generated by sets of the form $\{\omega \in \Omega : X(\omega) \geq a\}$. $\forall a \in \mathbb{R}$.

If X and Y have density functions then equivalently X and Y independent if $f_{XY}(x,y) = f_X(x) f_Y(y)$.

$F_X(x) = \mathbb{P}(X \leq x)$
↳ CDF

Ex $X, \cancel{Y}$ positive integer valued and independent with

$$\mathbb{P}(X=n) = \frac{n^{-2}}{\sum_{k} k^{-2}} \qquad n=1,2,\ldots$$

Define $E_2 = \{X \text{ is divisible by } 2\}$, $E_3 = \{X \text{ is divisible by } 3\}$.

Want to show that E_2 and E_3 are independent events

i.e. $\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(E_2)\,\mathbb{P}(E_3)$

$$\mathbb{P}(E_2) = \mathbb{P}(\{X=2\} \cup \{X=4\} \cup \ldots) = \sum_{n=1}^{\infty} \mathbb{P}(X=2n)$$

$$= \sum_{n=1}^{\infty} \frac{(2n)^{-2}}{\sum_{k} k^{-2}} = \frac{1}{4} \frac{\sum_{n=1}^{\infty} n^{-2}}{\sum_{k=1}^{\infty} k^{-2}} = \frac{1}{4}.$$

$$\mathbb{P}(E_3) = \mathbb{P}(X \text{ is divisible by } 3) = \sum_{n=1}^{\infty} \mathbb{P}(X = 3n)$$

$$= \sum_{n=1}^{\infty} \frac{(3n)^{-2}}{\sum_{k} k^{-2}} = \frac{1}{9}.$$

$$\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(X \text{ is divisible by } 6). = \sum_{n=1}^{\infty} \mathbb{P}(X = 6n)$$

$$= \sum_{n=1}^{\infty} \frac{(6n)^{-2}}{\sum_{k} k^{-2}} = \frac{1}{36}.$$

$$\therefore \mathbb{P}(E_2)\mathbb{P}(E_3) = \mathbb{P}(E_2 \cap E_3).$$