Problem 1.

Compute $\mathrm{Var}\left(\frac{1}{400} \sum_{j=1}^{400} \left(\frac{W(t_j) - W(t_{j-1})}{\sqrt{0.01}} \right)^2 \right)$

where $t_j = \frac{j}{100}$, $j = 0, 1, \ldots 400$

Solution: Let $X_j = \frac{W(t_j) - W(t_{j-1})}{\sqrt{0.01}}$,

then X_j is standard normal ~~R.~~ variable.

this is because $W(t_j) - W(t_{j-1}) \sim N(0, t_j - t_{j-1})$

$= N(0, \frac{1}{100})$

Recall that the moment generating function (m.g.f) of a standard normal r.v is:

$$f(u) = \mathbb{E}[e^{uX}] \stackrel{?}{=} e^{\frac{1}{2}u^2}$$

$$f'(0) \rightarrow \mathbb{E}X$$

$$f''(0) \rightarrow \mathbb{E}X^2$$

$$\vdots$$

$$f'(u) = u e^{\frac{1}{2}u^2} \qquad f''(u) = (1+u^2) e^{\frac{1}{2}u^2}$$

$$f'''(u) = (3u+u^3) e^{\frac{1}{2}u^2} \qquad f^{(4)}(u) = (3+6u^2+u^4) e^{\frac{1}{2}u^2}$$

Then $\mathbb{E}[X_j^4] = f^{(4)}(0) = (3+6\cdot 0^2 + 0^4) e^{\frac{1}{2}\cdot 0^2}$

$$= 3$$

$$\Rightarrow Var(X_j^2) = \mathbb{E}[X_j^4] - (\mathbb{E}X_j^2)^2$$

$$= 3 - 1^2$$

$$= 2$$

$$\mathrm{Var}\left(\frac{1}{400}\sum_{j=1}^{400} X_j^2\right) = \frac{1}{400^2}\sum_{j=1}^{400} \mathrm{Var}(X_j^2)$$

$$= \frac{1}{400^2} \cdot 2 \cdot 400$$

$$= \frac{1}{200}$$

Std. Dev is $\sqrt{\frac{1}{200}} \approx 0.07$.

~~Question 2:~~ Problem 2:

Ornstein-Uhlenbeck process. (OU process)

Find the distribution of $X(t)$ which satisfies:

$$dX(t) = K(\theta - X(t))\,dt + \sigma\, dW(t)$$

$K>0$, θ and σ are constants, $W(t)$ is a B.M.

We want to get rid of the $-KX(t)\,dt$ on the r.h.s

we introduce "integrating factor" e^{Kt} and then

compute $d(e^{Kt}X(t))$

Let $f(t,x) = e^{kt}x$, then $f(t,X(t)) = e^{kt}X(t)$

So $d(e^{kt}X(t))\cancel{x} = df(t,X(t))$, we use Ito's lemma to calculate $df(t,X(t))$:

$$df(t,X(t)) = f_t(t,X(t))\,dt + f_x(t,X(t))\,dX(t) + \frac{1}{2} f_{xx}(t,X(t))\,(dX(t))^2 \qquad (*)$$

$$f_t := \frac{\partial f}{\partial t}\cancel{x}(t,x) = \frac{\partial\, \cancel{e^{kt}x}\, e^{kt}x}{\partial t} = ke^{kt}x$$

$$f_x = \frac{\partial f}{\partial x}(t,x) = e^{kt}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}(t,x) = \frac{\partial f_x}{\partial x} = \frac{\partial e^{kt}}{\partial x} = 0$$

Plug these in $(*)$ and substitue $f(t,x)$ with

$f(t, X(t))$:

$$d f(t, X(t)) = k e^{kt} X(t)\, dt + e^{kt} dX(t) + \frac{1}{2} \cdot 0 \cdot (dX(t))^2$$

$$= k e^{kt} X(t) dt + e^{kt}(k(\theta - X(t))dt + \sigma dW(t))$$

$$= k\theta e^{kt} dt + \sigma e^{kt} dW(t)$$

$$\Rightarrow \quad d\, e^{kt} X(t) = k\theta e^{kt}\, dt + \sigma e^{kt}\, dW(t)$$

Integrate both sides from 0 to t:

$$\int_0^t : \quad e^{kt} X(t) - e^{k\cdot 0} X(0) = \int_0^t k\theta e^{ks}\, ds + \int_0^t \sigma e^{ks}\, dW(s)$$

$$\Rightarrow \quad e^{kt} X(t) = X(0) + \cancel{\theta(1-e^{-kt})}\ \theta e^{kt}(1-e^{-kt}) + \int_0^t \sigma e^{ks}\, dW(s)$$

Multiply e^{-kt} on both sides:

$$X(t) = e^{-kt}X(0) + \theta(1-e^{-kt}) + \sigma e^{-kt}\underbrace{\int_0^t e^{ks}\,dW(s)}$$

$\int_0^t e^{ks}\,dW(s)$ is a stochastic integral with nonrandom integrand., if $\Delta(t)$ is a nonrandom function, then $\int_0^t \Delta(s)\,dW(s) \sim N\left(0, \int_0^t \Delta(s)^2\,ds\right)$

So we know $\int_0^t e^{ks}\,dW(s) \sim N\left(0, \int_0^t e^{2ks}\,ds\right)$

$$= N\left(0, \frac{1}{2k}\left(e^{2kt}-1\right)\right)$$

$$\Rightarrow \quad \sigma e^{-kt} \int_0^t e^{ks} dW(s) \sim N\left(0, \frac{\sigma^2 e^{-2kt}}{2k}\left(e^{2kt}-1\right)\right)$$

$$= N\left(0, \frac{\sigma^2}{2k}\left(1-e^{-2kt}\right)\right)$$

$$\Rightarrow \quad X(t) \sim N\left(e^{-kt}X(0)+\theta\left(1-e^{-kt}\right), \frac{\sigma^2}{2k}\left(1-e^{-2kt}\right)\right)$$