Last time:

↪ Model for stock prices: $dS = \alpha S\,dt + \underbrace{\sigma S\,dW}_{\text{random flutations.}}$

$dW \to$ ~~sor~~ infintesmally small incremnt of BM.

Prob: $(\Omega, \mathcal{G}, P)$

Ω ← sample space

$\mathcal{G}$ ← σ-alg (events of whose prob are known.)

P ← prob measure.

$A \in \mathcal{G} \to A \subseteq \Omega$. (called an event).

$P(A) \in [0,1]$ rep prob that A occurs.

Random Variables:

Disc prob: A random variable is a function

$$X : \Omega \longrightarrow \mathbb{R}$$

(ie X is a fn, domain Ω, target $\mathbb{R}$.)

Q: "Is X true"

$\boxed{\text{Q: What is the prob } X \text{ is true?}}$

Event that X is true: $\{\omega \in \Omega \mid X(\omega) > 0\}$

Notation: $\{\omega \in \Omega \mid X(\omega) > 0\} = \{X > 0\} \overset{\text{def}}{=} A.$

Prob X is +ve is $P(A)$.

Only makes sense if $A \in \mathcal{G}$

<u>Def:</u> A random variable is a function.

$$X : \Omega \longrightarrow \mathbb{R} \quad \text{such that.}$$

for every $\alpha \in \mathbb{R}$, the event $\{X \le \alpha\} \in \mathcal{G}$.

(aka a $\mathcal{G}$-measurable R.V.).

($\Rightarrow$ events of the form $\{X > \alpha\}$, $\{X > \alpha \;\&\; X \le \beta\}$, $\{X \ge \alpha\}$, etc are all elements of $\mathcal{G}$.

Remark: If X, Y are two RV's;

then $X+Y$, $X \vee Y$, $X \wedge Y$, Ye^X ... etc

are all also $\mathcal{G}$-meas RV's.

Eg: Suppose $A \in \mathcal{G}$ is some event. $(A \subseteq \Omega)$.

Define $X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$

(Called the indicator function of A. Notation $\mathbb{1}_A =$).

Check. X is a RV:

Pick $\alpha \in \mathbb{R}$, $\quad \{X \leq \alpha\} = \begin{cases} \Omega & \alpha \geq 1 \\ A^c & \alpha \in [0,1) \\ \phi & \alpha < 0 \end{cases}$

Note Ω, ϕ & $A^c \in \mathcal{G}$ $\Rightarrow$ X is $\mathcal{G}$ meas R.V.

Expectations: $X \to$ R.V.

$$EX = \text{Expectation of } X.$$

$$= \text{Lebesgue integral of } X \text{ wrt } P$$

$$= \int_\Omega X \, dP = \int_\Omega X(\omega) \, dP(\omega).$$

Suppose X only takes on finitely many values $a_1, \dots a_N$.

Let $A_i = \{X = a_i\}$. $\left(\Omega = \underset{\substack{\uparrow \\ \text{disj}}}{\cup} A_i\right)$.

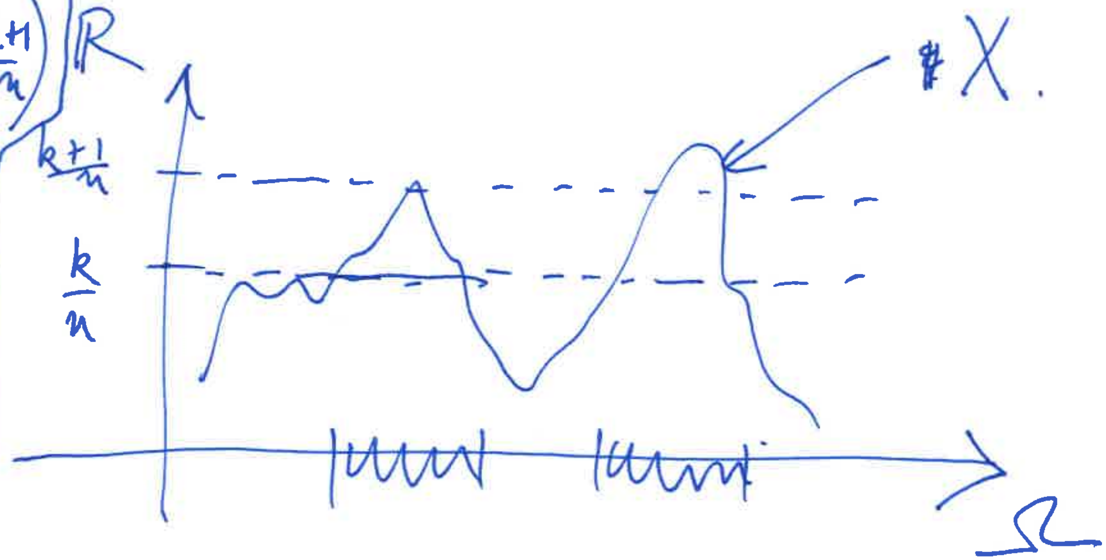
(Such RV's are called simple).

$$EX = \sum_{i=1}^{N} a_i P(X = a_i) = \sum_{i=1}^{N} a_i P(A_i).$$

In ~~Great~~ General $\longrightarrow$ approximate.

$$EX = \lim_{n\to\infty} E\left(\sum_{-n^2}^{n^2-1} \frac{k}{n} \mathbb{1}_{\left\{\frac{k}{n} \le X < \frac{k+1}{n}\right\}}\right).$$

$$= \lim_{n\to\infty} \sum_{-n^2}^{n^2-1} \frac{k}{n} P\left(\frac{k}{n} \le X < \frac{k+1}{n}\right)$$



Properties of Expectations.

① ~~Eg~~ X, Y R.V's. $\alpha \in \mathbb{R}$.

$$E(X + \alpha Y) = EX + \alpha EY.$$

② Positivity: If $X \geq 0$ $\boxed{\text{almost surely}}$,

then $EX \geq 0$

(Eg. Here if $X \leq Y$ almost surely $\Rightarrow EX \leq EY$.)

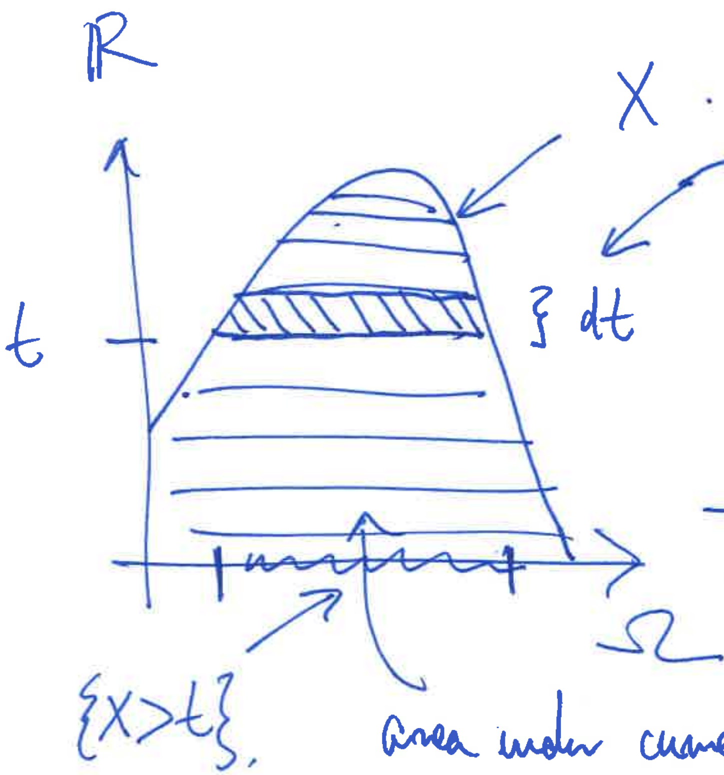
Note: $X \geq 0$ almost surely

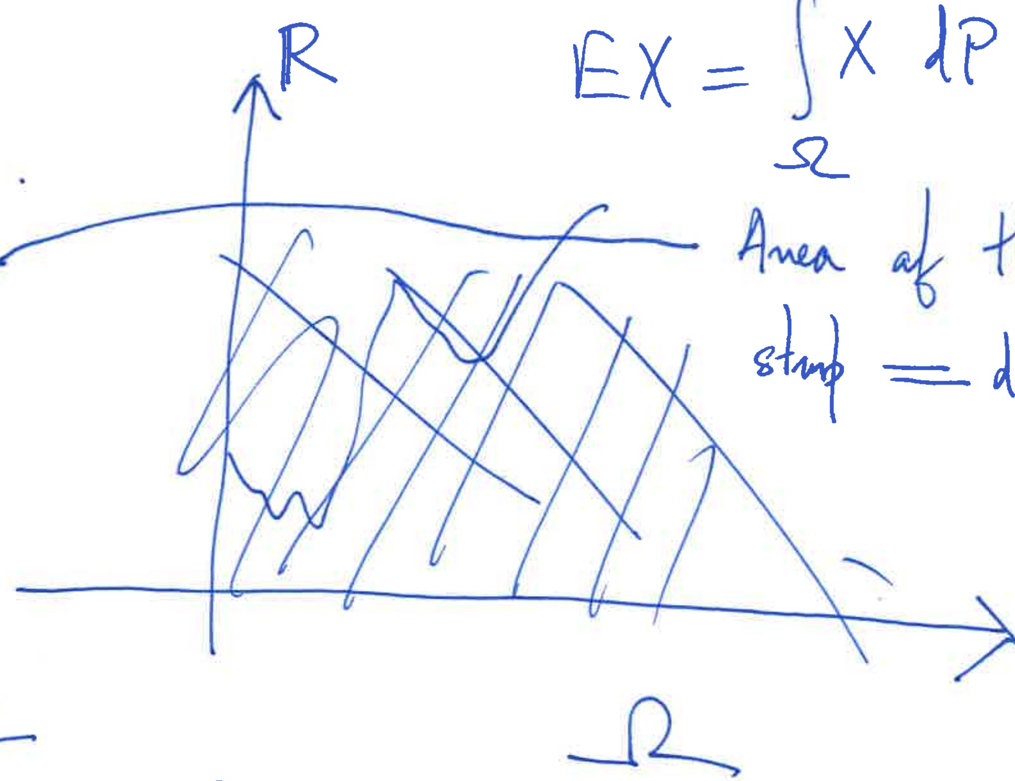
means $P(X \geq 0) = 1$.

③ Layer cake formula:

If $X \geq 0$ a.s. then

$$EX = \int_0^\infty P(X \geq t)\, dt .$$





$$EX = \int_\Omega X\, dP .$$

Area of the shaded strip $= dt \cdot P(X>t)$

area under curve $= EX$.

④ Lazy Stat / Unconscious stat formula

Say p is the prob density fn of X.

~~f is a~~ $f: \mathbb{R} \to \mathbb{R}$ is a nice fn.

Then $$E f(X) = \int_{-\infty}^{\infty} f(x) \, p(x) \, dx$$

Last time: B.M. is a $\boxed{\text{cts process}}$ with stationary ind inc.

Process with cts traj:

↳ For every $t \geq 0$, $X(t)$ is a R.V.

__Process:__ ~~X~~ A function $X : \Omega \times [0,\infty) \longrightarrow \mathbb{R}$

is a stochastic process.

X is a fn of 2 vars ① $\omega \in \Omega$.

② $t \in [0,\infty)$.

~~Xt~~ (Fix t: $X(\omega, t)$ is a fn of ω

Want for every $t \geq 0$, $X(\omega, t)$ (viewed as a fn of ω)

to be a R.V.

For every fixed $\omega \in \Omega$, $X(\omega, t)$ viewed as a fn of $\underline{\underline{t}}$

is the trajectory of X. (Cts process want all traj to be cts).

Independence:

① $A, B \in \mathcal{G}$ (A & B are events).

We say A & B are independent if $\underline{P(A \cap B) = P(A)\,P(B)}$.

$$\left(P(A \mid B) = P(A) \right)$$

$$\parallel$$

$$\frac{P(A \cap B)}{P(B)}.$$

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② $A_1, A_2, \cdots A_N$ are N events in $\mathcal{G}$.

We say $\{A_1, \ldots A_N\}$ are ind if.

$$P\left(\bigcap_{k=1}^{d} A_{i_k} \right) = \prod_{k=1}^{d} P\left(A_{i_k} \right)$$

X, Y two RV's: X & Y are independent.

$$\cancel{E f(X) g(Y) = \left(E f(X)\right)\left(E g(Y)\right)}$$

$\boxed{\text{Every event observable through } X}$ is <u>independent</u> of Every event observable through Y.

<u>Def:</u> Let X be a RV.

Define $\sigma(X) = \sigma$ alg generated by X

$= \{$all events that can be observed through $X\}$.

$= \sigma$ alg generated by $\{X \leq \alpha \mid \alpha \in \mathbb{R}\}$.

i.e. $\sigma(X)$ is a σ-alg.

& $\{X \leq \alpha\} \in \sigma(X)$ for every $\alpha \in \mathbb{R}$.

($\Rightarrow$ $\{X \in [\alpha, \beta)\}$, $\{X > \alpha\}$, $\{\sin(X) > e^{3X}\}$ etc.

are all elents of $\sigma(X)$).

Eg: $\Omega = \{1, \cdots 52\}$ (cards).

Game: $\rightarrow$ Win \$1 if red card is drawn $\{1, \cdots 26\}$.

$\rightarrow$ Lose \$1 " black " " " $\{27, \cdots 52\}$.

(all cards are drawn with equal prob).

$\mathcal{G} = \mathcal{P}(\Omega)$ = all subsets of Ω.

$A \in \mathcal{G}$, $\quad P(A) = \dfrac{\#\text{ of elements in } A}{52}$.

$R = \{1, \cdots 26\}$ $\qquad$ $B = \{27, \cdots 52\}$

X = outcome of my game $= \mathbb{1}_R - \mathbb{1}_B$

Q: $\sigma(X)$ = all info obtained by obs'g X

Just by obs'ing X: Can deduce

$P(\emptyset) = 0$

$P(R) = 1/2$

$P(B) = 1/2$.

$P(\Omega) = 1$

Note $\sigma(X)$ = σ alg gen by $\{\{X \le \alpha\} \mid \alpha \in \mathbb{R}\}$.

$= \{\phi, R, B, \Omega\}$.

Intuition in prob: think of $\sigma(X)$ = all info that can be obtained by obsving X.

Def: We say two RV's X & Y are ind if every event in $\sigma(X)$ is ind of every event in $\sigma(Y)$.

Tests: Let $X_1, \dots X_N$ be N RV's.

Thm ① $X_1, X_2, \dots X_N$ are independent

$\iff$ ② For every $\alpha_1, \alpha_2 \dots \in \mathbb{R}$ we have

$$P\left(\bigcap_{j=1}^{N} \{X_j \le \alpha_j\}\right) = \prod P(X_j \le \alpha_j)$$

③ For every coll of boundd cts fn's $f_1, f_2 \dots f_N$,

$$E\left(\prod_{j=1}^{N} f_j(X_j)\right) = \prod_{j=1}^{N} E f_j(X_j).$$

④ For any $t_1, \dots t_N \in \mathbb{R}$,

$$\text{Joint mgf} \rightarrow E \exp\left(\sum_{j=1}^{N} t_j X_j\right) = \prod_{j=1}^{N} \underline{E} \cdot \exp(t_j X_j) \leftarrow \text{mgf of } X_j$$