944 STOCHASTIC CALCULUS I

MIKE ~~HE~~P

Office Hours: 10 – 11 AM (Tue NY Zoom)

11 – 12 noon (Tue Pitt in WEH 6121 (me)).

10 – 11 am (Wed Pitt, David 7211)

11 – 12 noon (Wed NY, Canvas, David)

Midterm 1: Thu Feb 6 ← 25%

HW: 10%.

Final: 60%.

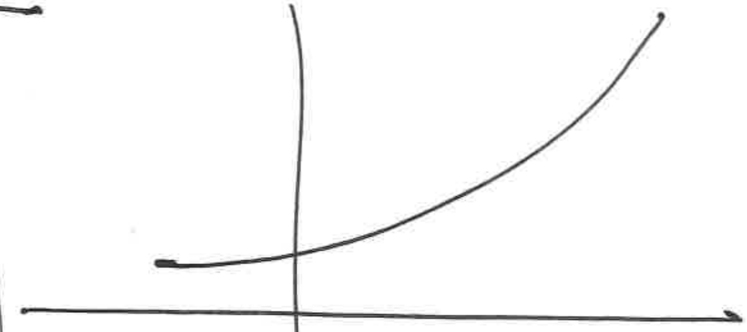
Attendance: 5%

Recitation Friday 9:30 AM

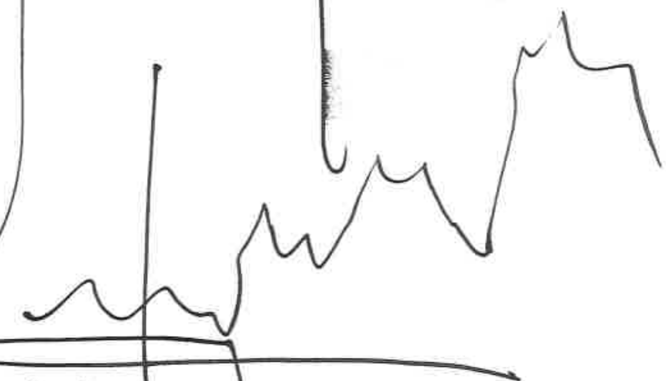
① Introduce **Stochastic Calculus**.

$$\cancel{dS_t = \alpha S \, dt}$$

$$\frac{dS_t}{dt} = \alpha S$$



Stock prices:



Model: $dS = \alpha S \, dt +$ "Random fluctuations"

② Black Scholes formula. (Continuous time).

↳ Derivative pricing.

European Call option.

Stock $\rightarrow$ Say stock price is x.

European Call, on maturity T, strike K.

Value of the call: $\overset{c(t,x)}{\cancel{c(x,t)}} = ?$

Clearly $\overset{c(T,x)}{\cancel{c(x,T)}} = \underline{\underline{(x-K)^+}}$

When $t < T$:

B.S. Formula: $c(t,x) = x N(d_+(T-t,x)) - K e^{-r(T-t)} N(d_-(T-t,x))$

where: $d_\pm(\tau, x) = \dfrac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\dfrac{x}{K}\right) + \left(r \pm \dfrac{\sigma^2}{2}\right)\tau\right)$.

$\sigma \longrightarrow$ "volatilatility" (% volatility).

$r \longrightarrow$ interest rate.

$$dS = \alpha S\,dt + \sigma S\,dW$$

α: mean return rate.

σ: vol.

dW: Brownian motion increment. (IOU).

Price option: Replication.

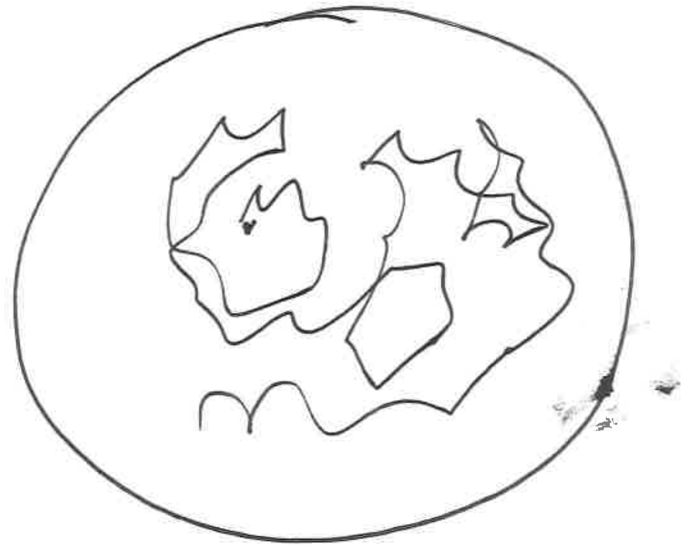
$\boxed{\text{① Stock..}}$

② M.M. act with interest rate r.

Risk neutral measures $\longrightarrow$ <u>Price anything</u>.

Brownian Motion: "Continuous time Random Walk".

Discrete time random walk.



Let $\xi_1, \xi_2, \ldots$

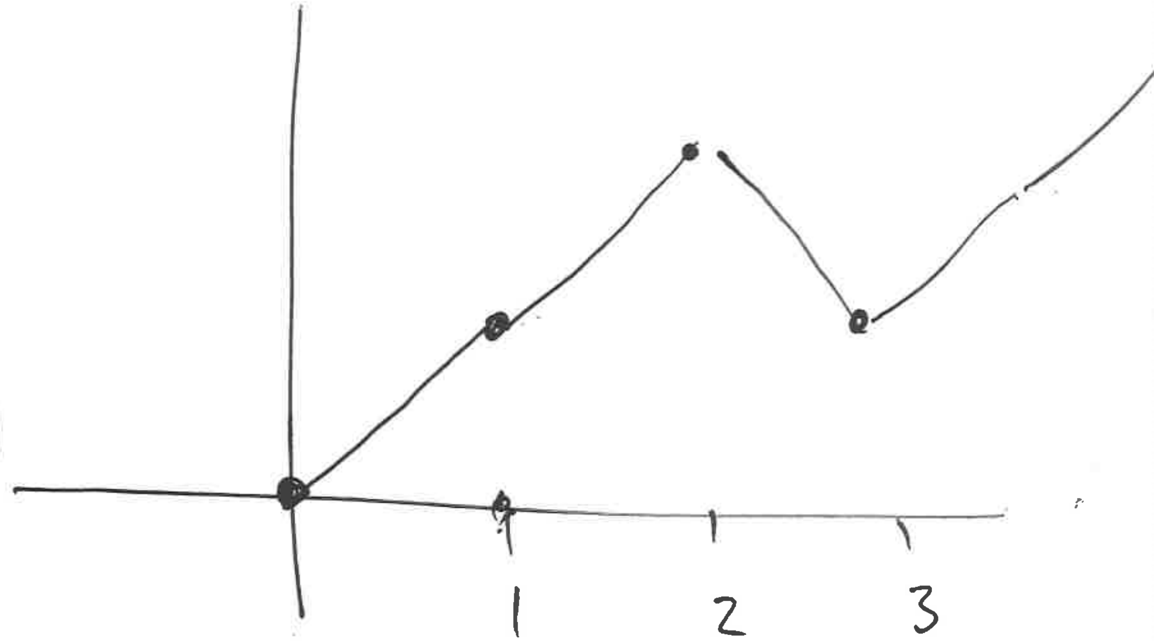
sequence of iid RV's.

Let $\cancel{S(t)}$ $S(0) = 0$

$$S(t) = S(n) + (t-n)\,\xi_{n+1} \quad \text{if } t \in (n, n+1].$$

Scale this:

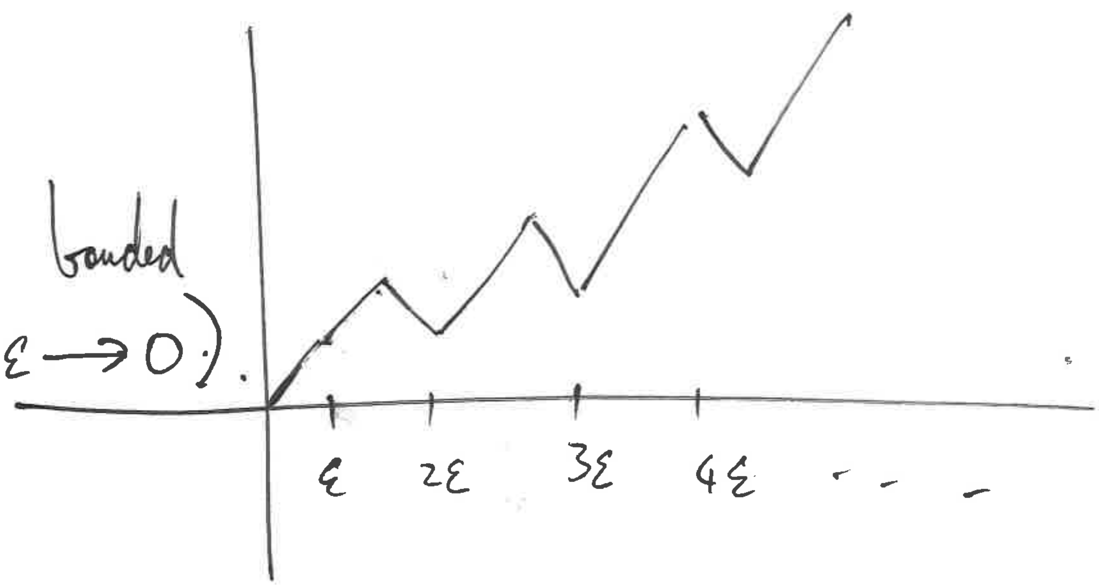
Let $\varepsilon > 0$



Let $S_\varepsilon(t) = \sqrt{\varepsilon}\, S\left(\frac{t}{\varepsilon}\right)$.

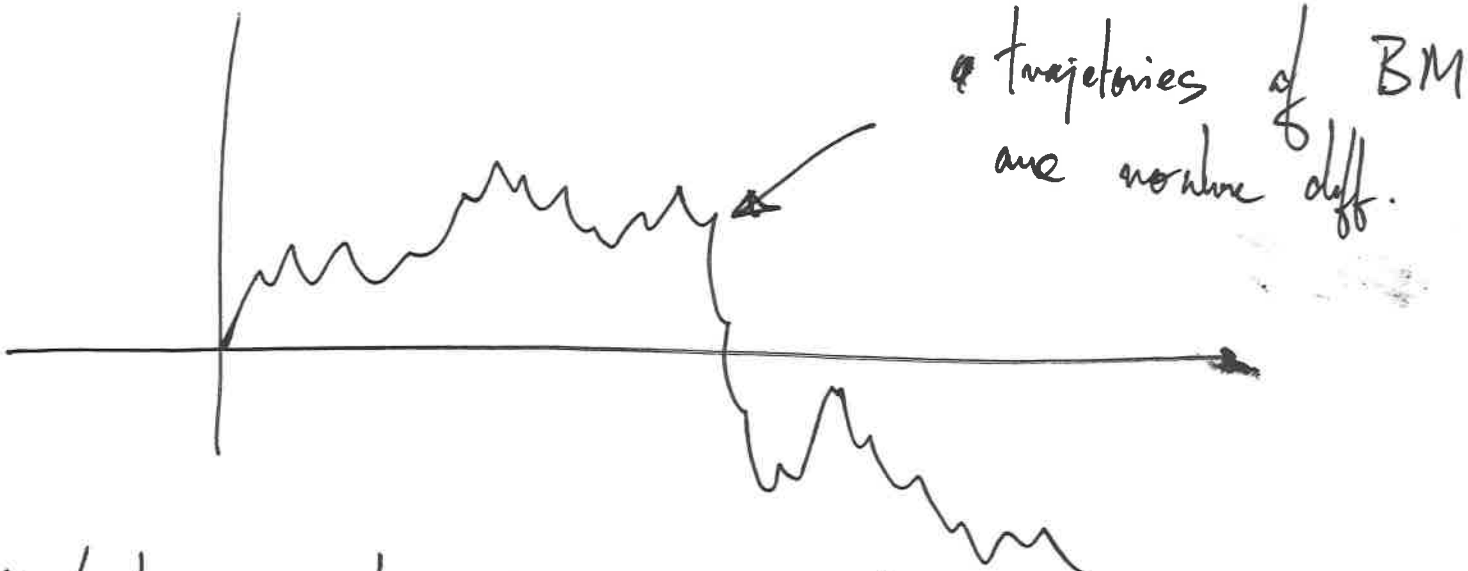
$S_\varepsilon(1) = \sqrt{\varepsilon}\,.\, S\left(\frac{1}{\varepsilon}\right)$

(Compute $\mathrm{Var}\left(S_\varepsilon(t)\right)$ remains bounded as $\varepsilon \to 0$).



Theorem: The process $S_\varepsilon(t)$ converges as $\varepsilon \longrightarrow 0$.

The limiting process is called Brownian motion.



More Useful Definition of Brownian Motion:

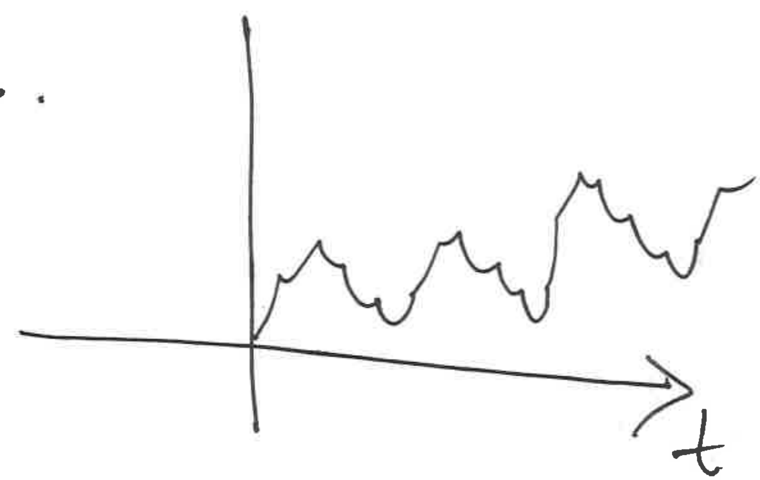
Def: B.M. is a cts process that has <u>stationary</u> <u>independent</u> <u>increments</u>.

(1) A process (stochastic process) is
a collection of RVs. one for each $t \geq 0$.
I.e. $\{X(t) \mid t \geq 0\}$ is a S.P.
if each $X(t)$ is a R.V.

(2) A trajectory of a process is one outcome of each
R.V. $X(t)$, viewed as a fn of t.

(3) Cts process: trajectories are cts.

$$\lim_{s \to t} X(s) = X(t)$$



④ Stationary increments:

For every $h \geq 0$, $X(t+h) - X(t)$ has the same distribution for every $t \geq 0$.

⑤ Independent increments:

For every $0 \leq t_0 \leq t_1 < t_2 \cdots t_N$.

the RV's. $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, ...

are all independent.

$$S_\varepsilon(t) - S_\varepsilon(s) \sim \left(\sum_{i=1}^{(t-s)/\varepsilon} \xi_i \right) \cdot \sqrt{\varepsilon}$$

(Assume t & s are integer multiples of ε)

Note as $\varepsilon \to 0$, $$\sqrt{\varepsilon} \sum_{i=1}^{\frac{t-s}{\varepsilon}} \xi_i \xrightarrow{\varepsilon \to 0} N(0, t-s).$$

I.e. $$S_\varepsilon(t+h) - S_\varepsilon(t) \xrightarrow[\text{dist}]{\varepsilon \to 0} N(0, h)$$

independent of t.

(Stationary increments).

Also get independent increments similarly.

Def 3 (Brownian Motion).

~~letters~~.

Brownian Motion is a cts process W such that.

1. W has independent increments.
2. for $s < t$, $W(t) - W(s) \sim N(0, \sigma^2(t-s))$.

Std B.M. Choose $\sigma = 1$.

Probability: $(\Omega, \mathcal{G}, P)$

Ω: sample space (some set).

$\mathcal{G}$: σ- algebra. (events you know the probability of.)

P: probability measure.

σ – algebra:

Event: $\boxed{\text{subsets of } \Omega.}$

σ – alg: A collection of events.

We say $\mathcal{G}$ is a σ – alg if.

1. $\mathcal{G}$ is a non-empty collection of subets of Ω.
2. $\mathcal{G}$ is closed under complints.
   (i.e. If $A \in \mathcal{G}$, then $A^c \in \mathcal{G}$).
3. ctable unions: If $A_1, A_2, \cdots \in \mathcal{G}$, then

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \in \mathcal{G}.$$

4. $\phi \in \mathcal{G}$, $\Omega \in \mathcal{G}$

5. If $A_1, A_2, \dots \in \mathcal{G}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{G}$.

6. If $A, B \in \mathcal{G}$, then $A - B \in \mathcal{G}$.

(Note $A - B = A \cap B^c$)

<u>Def</u>: P is a probability measure if. (Intution: $P(A)$ = prob of event A occuring).

1. For every $A \in \mathcal{G}$, $P(A) \in [0, 1]$.

   and $P(\Omega) = 1$. (& $P(\phi) = 0$).

& ② (Countable additivity).

If $A_1, A_2, \ldots$ are pair-wise disjoint.

then $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Note ③ $$P(A^c) = 1 - P(A).$$

[Note $P(\Omega) = 1$
$$\begin{array}{c} P(\Omega) = 1 \\ \| \\ P(A \cup A^c) = P(A) + P(A^c) \end{array}$$
]

④ If $A, B \in \mathcal{G}$, $$P(A - B) = P(A) - P(A \cap B).$$

If ~~$A \subseteq B$~~ $B \subseteq A$ $\Rightarrow$ $$P(A - B) = P(A) - P(B).$$

(5a) If $A_1 \subseteq A_2 \subseteq \cdots$

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i=\infty} P(A_i)$.

(5b) If $A_1 \supseteq A_2 \supseteq \cdots$

then $P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i)$.