MSCF 944 Homework

The homework policy on the class website will be *strictly* enforced.

Assignment 1 (assigned 2020-01-16, due 2020-01-23).

1. Let X be a random variable. The *moment generating function* of X (denoted by M_X) is defined by $M_X(t) = \boldsymbol{E} e^{tX}$. It is certainly possible that $M_X(t) = \infty$ for some t, but for this problem we assume that $M_X(t)$ is finite in some small interval containing 0.
   (a) If M_X is infinitely differentiable, find a relationship between the n^{th} derivative of M_X and $\boldsymbol{E}X^n$. Provide some reasoning.
   (b) If $X \sim \text{Exp}(\lambda)$, compute $M_X(t)$ for every $t \in \mathbb{R}$.
   (c) If $X \sim N(\mu, \sigma^2)$ compute $M_X(t)$ for all $t \in \mathbb{R}$.
2. Let X, Y be two random variables. Recall the covariance $\operatorname{cov}(X, Y)$ is defined by $\operatorname{cov}(X, Y) = \boldsymbol{E}(XY) - (\boldsymbol{E}X)(\boldsymbol{E}Y)$. Two random variables are called uncorrelated if $\operatorname{cov}(X, Y) = 0$.
   (a) If $a, b, c, d \in \mathbb{R}$, compute $\operatorname{cov}(aX + b, cY + d)$ in terms of $\operatorname{cov}(X, Y)$ and a, b, c and d.
   (b) Show that independent random variables are uncorrelated.
   (c) Show by example that uncorrelated random variables need not be independent.
   (d) If (X, Y) is jointly normal and X and Y are uncorrelated, must X and Y be independent? Justify. [Do more than simply quote a theorem.]
3. (a) *(Chebychev's inequlity)* For any $p, \lambda > 0$, prove $\boldsymbol{P}(X > \lambda) \leqslant \boldsymbol{E}(|X|^p)/\lambda^p$. [HINT: For $p = 1$, verify and use the fact that $\lambda \boldsymbol{1}_{\{X > \lambda\}} \leqslant |X|$.]
   (b) *(Jensen's inequality)* If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, and X is a random variable, show that $\varphi(\boldsymbol{E}X) \leqslant \boldsymbol{E}\varphi(X)$. [HINT: Use the fact that convex functions are always *above* their tangent. Namely, for any $a \in \mathbb{R}$, we have $\varphi(a) + (X - a)\varphi'(a) \leqslant \varphi(X)$. Choose a correctly and use positivity. If this hint isn't sufficient, this should be done in the text and in the self study videos.]
4. (a) If X is a continuous random variable with density p, we know $\boldsymbol{E}X = \int_{-\infty}^{\infty} x p(x)\, dx$. If X is also nonnegative, use the above formula to derive the *layer cake formula*
   $$\boldsymbol{E}X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt\,,$$
   in this special case.
   (b) Let X be a nonnegative random variable (which may or may not have a density), and let φ be a differentiable, nonnegative, increasing function with $\varphi(0) = 0$. Use the layer cake formula to show that
   $$\boldsymbol{E}\varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt\,.$$
   (c) Is this formula still valid if $\varphi(0) \neq 0$?

Let $\Omega = [0, 1)$, and let $\mathcal{B}$ be the *Borel* σ-algebra (i.e. the σ-algebra generated by all open intervals in Ω). It can be shown that there exists a measure $\boldsymbol{P}$ on $(\Omega, \mathcal{B})$ such that $\boldsymbol{P}((a, b)) = b - a$ for every $a, b \in \Omega$ with $a \leqslant b$. Moreover, under this measure, the expectation operator is simply the Riemann integral, when it exists. Namely, given a random variable $X \colon \Omega \to \mathbb{R}$, we have $\boldsymbol{E}X = \int_0^1 X(y)\, dy$, provided X is Riemann integrable. This measure is known as the *Lebesgue measure*, and is extremely important. Its rigorous construction, however, isn't straightforward.

5. Let $H \colon \mathbb{R} \to \mathbb{R}$ be defined by $H(x) = 0$ if $\lfloor x \rfloor$ is even, and $H(x) = 1$ otherwise. For $n \in \{1, 2, \dots\}$, define $X_n \colon \Omega \to \mathbb{R}$ by $X_n(y) = H(2^n y)$.
   (a) Show that X_n is a random variable for every n.
   (b) Given $\alpha \in \mathbb{R}$, and $m \neq n$ compute $\boldsymbol{P}(X_m < \alpha)\boldsymbol{P}(X_n < \alpha)$.
   (c) Given $\alpha \in \mathbb{R}$, and $m \neq n$ compute $\boldsymbol{P}(X_m < \alpha \;\&\; X_n < \alpha)$.
   (d) Compute $\boldsymbol{E}X_n$, $\boldsymbol{E}X_n^2$ and $\boldsymbol{E}X_m X_n$ when $m \neq n$.

Assignment 2 (assigned 2020-01-23, due 2020-01-30).

Unless otherwise noted, $\{\mathcal{F}_t\}$ is the Brownian filtration, and W is a standard Brownian motion. In all problems that ask you to simply compute something, you should also explain how you arrived at the answer and not simply state the answer.

1. Let $\Omega = \{HH, HT, TH, TT\}$ be the sample space corresponding to two tosses of a coin. Let X and Y be the number of heads on the first and second tosses respectively. That is

$$X(HH) = X(HT) = 1, \quad X(TH) = X(TT) = 0\,,$$
$$Y(HH) = Y(TH) = 1, \quad Y(HT) = Y(TT) = 0\,.$$

   (a) Enumerate $\sigma(X)$, and $\sigma(Y)$ explicitly.

   (b) Define a probability measure $\boldsymbol{P}$ by

$$\boldsymbol{P}\{HH\} = \frac{1}{12}, \quad \boldsymbol{P}\{HT\} = \frac{1}{6}, \quad \boldsymbol{P}\{TH\} = \frac{1}{4}, \quad \boldsymbol{P}\{TT\} = \frac{1}{2}.$$

   Are X and Y independent under $\boldsymbol{P}$? Justify your answer.

   (c) Consider the probability measure $\tilde{\boldsymbol{P}}$ defined by

$$\tilde{\boldsymbol{P}}\{HH\} = \frac{1}{12}, \quad \tilde{\boldsymbol{P}}\{HT\} = \frac{1}{6}, \quad \tilde{\boldsymbol{P}}\{TH\} = \frac{3}{8}, \quad \tilde{\boldsymbol{P}}\{TT\} = \frac{3}{8}.$$

   Are X and Y independent under $\tilde{\boldsymbol{P}}$? Justify your answer.

2. Consider the sample space

$$\Omega \stackrel{\text{def}}{=} \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}\,.$$

For $p \in (0,1)$ let $\boldsymbol{P}$ be the probability measure on Ω correspond to independent coin tosses that yield heads with probability p, and tails with probability q. In other words, $\boldsymbol{P}(HHH) = p^3$, $\boldsymbol{P}(HHT) = p^2 q$, and so on. Consider a three-period stock price model defined as follows: Let $S_0(\omega) = 4$, and for $n \in \{1, 2, 3\}$ define

$$S_n(\omega) = \begin{cases} 2S_{n-1}(\omega) & \text{if the } n^{\text{th}} \text{ letter in } \omega \text{ is } H\,, \\ \dfrac{S_{n-1}(\omega)}{2} & \text{otherwise}\,. \end{cases}$$

Note $\sigma(S_3)$ is generated by the atoms

$$C_1 \stackrel{\text{def}}{=} \{S_3 = 32\} = \{HHH\}\,, \qquad C_2 \stackrel{\text{def}}{=} \{S_3 = 8\} = \{HHT, HTH, THH\}\,,$$
$$C_3 \stackrel{\text{def}}{=} \{S_3 = 2\} = \{HTT, THT, TTH\}\,, \qquad C_4 \stackrel{\text{def}}{=} \{S_3 = 0.50\} = \{TTT\}\,.$$

Let $\boldsymbol{E}(S_2 \mid S_3) = \boldsymbol{E}(S_2 \mid \sigma(S_3))$ denote the conditional expectation of S_2 given $\sigma(S_3)$.

   (a) Find constants $c_1, \ldots, c_4$ so that $\boldsymbol{E}(S_2 \mid S_3) = \sum_{i=1}^4 c_i \mathbf{1}_{C_i}$. [Hint: By definition of conditional expectations, we know $\int_{C_1} \boldsymbol{E}(S_2 \mid S_3)\, d\boldsymbol{P} = \int_{C_1} S_2 \, d\boldsymbol{P}$. In this context, this translates to the relation $\sum_{\omega \in C_1} \boldsymbol{E}(S_2 \mid S_3)(\omega) \boldsymbol{P}(\omega) = \sum_{\omega \in C_1} S_2(\omega) \boldsymbol{P}(\omega)$. Use this to find c_1, and proceed similarly for $c_2, \ldots c_4$.]

   (b) Explicitly compute $\boldsymbol{E}\boldsymbol{E}(S_2 \mid S_3)$ and verify that it equals $\boldsymbol{E}S_2$.

3. If $s < t$, compute $\boldsymbol{E}(W(t)^3 \mid \mathcal{F}_s)$.

4. Let Y be a standard normal random variable, and let $K \in \mathbb{R}$.

   (a) For any $x \in \mathbb{R}$ let $g(x) \stackrel{\text{def}}{=} \boldsymbol{E}((e^{(x+Y)} - K)^+)$. Express g explicitly in terms of the cumulative normal distribution function

$$N(d) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}\xi^2} \, d\xi,$$

   for two different values of d. [Your answer will look something like the Black-Scholes formula.]

   (b) Suppose now X is another standard normal random variable that is independent of Y. Compute $\boldsymbol{E}\big((e^{X+Y} - K)^+ \bigm| X\big)(\omega)$. [Even though the variable ω is usually suppressed from all formulae, include it explicitly in this problem for clarity. Recall that $\boldsymbol{E}\big((e^{X+Y} - K)^+ \bigm| X\big)$ is shorthand for $\boldsymbol{E}\big((e^{X+Y} - K)^+ \bigm| \sigma(X)\big)$.]

5. Given $\lambda \in \mathbb{R}$, find α so that the process $M(t) = \exp(\lambda W(t) - \alpha t)$ is a martingale.

Assignment 3 (assigned 2020-01-30, due 2020-02-04).

In light of your midterm on 2/6, this homework is due Tuesday 2/4, *instead of Thursday 2/6. Solutions to this homework will post on Tuesday 2/5, and consequently this homework will not be accepted late.*

Unless otherwise noted, $\{\mathcal{F}_t\}$ is the Brownian filtration, and W is a standard Brownian motion. In all problems that ask you to simply compute something, you should also explain how you arrived at the answer and not simply state the answer.

1. Let $t_1 > 0$ and $\xi_0 \in \mathbb{R}$ be a $\mathcal{F}_0$-measurable random variable, and ξ_1 be a $\mathcal{F}_{t_1}$ measurable random variable. Let I, A be the process defined by

$$I(t) = \begin{cases} \xi_0 W(t) & t < t_1\,, \\ \xi_0 W(t_1) + \xi_1 (W(t) - W(t_1)) & t \geqslant t_1\,. \end{cases}$$

and

$$A(t) = \begin{cases} \xi_0^2 t & t < t_1\,, \\ \xi_0^2 t_1 + \xi_1^2 (t - t_1) & t \geqslant t_1\,. \end{cases}$$

Explicitly check that the processes I and $I^2 - A$ are martingales, and conclude that $[I, I] = A$.

Note: This is a special case of Chapter 3, Lemma 4.1 in the notes, and is an immediate consequence of the properties of Itô integrals. However, please do not use Lemma 4.1 or Theorem 4.2 here, and do this problem directly. This is the key idea behind the construction of the Itô integral (and the proofs of the above lemmas), and it is very helpful if you explicitly check this yourself explicitly.

Hint: I recommend you start by showing I is a martingale. For this you need to show $\boldsymbol{E}(I(t) \mid \mathcal{F}_s) = I(s)$. Split the analysis into three cases: $s < t < t_1$, $s < t_1 \leqslant t$ and $t_1 \leqslant s < t$, and use properties of conditional expectations you know. The same strategy can be used to show $I^2 - A$ is a martingale. Once you figure this out, the general case (stated in class) follows by the same idea and some technical suffering with summation indices.

2. (a) Suppose (X_1, X_2) is jointly Gaussian with $\boldsymbol{E}X_i = 0$, $\boldsymbol{E}X_i^2 = \sigma_i^2$, and $\boldsymbol{E}X_1X_2 = \rho$. Find $\boldsymbol{E}(X_1 \mid X_2)$ (recall from your previous homework that $\boldsymbol{E}(X_1 \mid X_2)$ is shorthand for $\boldsymbol{E}(X_1 \mid \sigma(X_2))$). Express your answer in the form $g(X_2)$, where g is some function you have an explicit formula for.

   Hint: Let $Y = X_1 - \alpha X_2$, and choose $\alpha \in \mathbb{R}$ so that $\boldsymbol{E}YX_2 = 0$. By the normal correlation theorem we know Y is independent of X_2. Now use the fact that $X_1 = Y + \alpha X_2$ to compute $\boldsymbol{E}(X_1 \mid X_2)$.

   (b) Use the previous part to compute $\boldsymbol{E}(W(s) \mid W(t))$ when $s < t$. [This was asked in a job interview.]

3. Let $\alpha, \sigma \in \mathbb{R}$ and define $S(t) = S(0) \exp\Big(\Big(\alpha - \dfrac{\sigma^2}{2}\Big)t + \sigma W(t)\Big)$.

   (a) Given a function $f \colon \mathbb{R} \to \mathbb{R}$, find a function $g \colon \mathbb{R} \to \mathbb{R}$ so that

$$\boldsymbol{E}\big(f(S(t)) \bigm| \mathcal{F}_s\big) = g(S(s))\,.$$

   Your formula for g will involve f and an integral involving the density of the normal distribution. [Hint: Let $Y = \exp((\alpha - \frac{\sigma^2}{2})(t-s) + \sigma(W(t) - W(s)))$, and note $S(t) = S(s)Y$ where $S(s)$ is $\mathcal{F}_s$ measurable and Y is independent of $\mathcal{F}_s$. Use this to compute $\boldsymbol{E}(f(S(s)Y) \mid \mathcal{F}_s)$.]

   (b) Find functions $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ so that

$$S(t) = S(0) + \int_0^t f(s, S(s))\, ds + \int_0^t g(s, S(s))\, dW(s)\,.$$

   Hint: Use the Itô formula to compute $dS(t) = S(0)\, d(\exp(\cdots))$. If you get the right answer you'll realize the importance of the process S to financial mathematics. The fact that I called it S and not X might have already given you a clue...

   (c) Using the previous part find all $\alpha \in \mathbb{R}$ for which S is a martingale?

   (d) Let $\mu(t) = \boldsymbol{E}S(t)$. Find a function h so that $\partial_t \mu(t) = h(t, \mu(t))$. [You can do this directly using the formula for S, of course. But it might be easier (and more instructive) to use your answer to part (b) instead.]

   (e) Find a function h so that $[S, S](t) = \displaystyle\int_0^t h(s, S(s))\, ds$.

In part (a) above, we observe that if we apply any function f to the process S at time t and condition it on $\mathcal{F}_s$, the whole history up to time s, we get something that only depends on $S(s)$ (the "state" at time s) and *not* anything before. This is called the Markov property. Explicitly, a process X is called *Markov* if for any function $f \colon \mathbb{R} \to \mathbb{R}$ and any $s < t$ we have $\boldsymbol{E}(f(X(t)) \mid \mathcal{F}_s) = g(X(s))$ for some function g.

   (f) Is Brownian motion a Markov process? Justify.

4. (a) Find functions f, g so that $W(t)^4 = \displaystyle\int_0^t f(s, W(s))\, ds + \int_0^t g(s, W(s))\, dW(s)$.

   (b) Compute $\boldsymbol{E}W(t)^4$ explicitly as a function of t.

   (c) Find a function h so that $[W^4, W^4](t) = \displaystyle\int_0^t h(s, W(s))\, ds$.

5. Determine whether the following identities are true or false, and justify your answer.

   (a) $e^{2t} \sin(2W(t)) = 2 \displaystyle\int_0^t e^{2s} \cos(2W(s))\, dW(s)$.

   (b) $|W(t)| = \displaystyle\int_0^t \operatorname{sign}(W(s))\, dW(s)$. [Recall $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 0$ if $x = 0$.]

Assignment 4 (assigned 2020-02-04, due never).

In light of your midterm on 2/6, this homework is optional. It will not be graded, and solutions will not be posted. All the problems are good exam practice, so I recommend trying them. Also, a few problems will make their way to your next homework.

Unless otherwise noted, $\{\mathcal{F}_t\}$ is the Brownian filtration, and W is a standard Brownian motion. In all problems that ask you to simply compute something, you should also explain how you arrived at the answer and not simply state the answer.

1. For each process X defined below explicitly find adapted processes b, σ such that for any $s < t$ we have

$$X(t) = X(s) + \int_s^t b(r)\,dr + \int_s^t \sigma(r)\,dW(r)$$

   (a) $X(t) = \dfrac{2t}{1 + 3W(t)^2}$.

   (b) $X(t) = (1 + 2tW(t))^{10}$

   (c) $X(t) = \ln\bigl(1 + 2W(t)^4\bigr)$

   (d) $X(t) = W(t) \displaystyle\int_0^{W(t)} \exp(-ts^2)\,ds$.

2. Determine if the following processes are martingales.

   (a) $X(t) = (W(t) + t)\exp(-W(t) - t/2)$.

   (b) $X(t) = \Bigl(W(t) + \dfrac{t^2}{2}\Bigr)\exp\Bigl(-\displaystyle\int_0^t s\,dW(s) - \dfrac{t^2}{2}\Bigr)$

   (c) $X(t) = \Bigl(W(t) + \displaystyle\int_0^t b(s)\,ds\Bigr)\exp\Bigl(-\displaystyle\int_0^t b(s)\,dW(s) - \dfrac{1}{2}\displaystyle\int_0^t b(s)^2\,ds\Bigr)$, where b is any adapted process.

   The third part above requires the use of the multidimensional Itô formula, or the product rule which I have not yet covered. These are special cases of the Girsanov theorem, which we will be important later, and we will revisit these problems after doing the Girsanov theorem.

Note that if X is a process with mean 0 independent increments (i.e. $X(t) - X(s)$ is independent of $\mathcal{F}_s^X$), then X must be a martingale with respect to the filtration generated by X. The converse is false. Here is a counter example.

3. Let $M(t) = \int_0^t W(s)\,dW(s)$.

   (a) For $s < t$, compute $\boldsymbol{E}\bigl((M(t) - M(s))^2 \bigm| \mathcal{F}_s\bigr)$.

   (b) Compute $\boldsymbol{E}(M(t) - M(s))^2 W(s)^2$ and $\boldsymbol{E}(M(t) - M(s))^2 \boldsymbol{E}W(s)^2$, and verify that they are not equal. Conclude $M(t) - M(s)$ is not independent of $\mathcal{F}_s$.

   (c) *(Unrelated)* Given $\lambda \in \mathbb{R}$ and $s < t$ show that

$$\boldsymbol{E}\bigl(e^{\lambda(M(t)-M(s))} \bigm| \mathcal{F}_s\bigr) = 1 + \frac{\lambda^2}{2}\int_s^t \boldsymbol{E}\bigl(e^{\lambda(M(r)-M(s))} W(r)^2 \bigm| \mathcal{F}_s\bigr)\,dr\,.$$

4. *(Itô and martingale representation theorems.)* Fix $T > 0$, and suppose $f = f(x)\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

   (a) We know that $\boldsymbol{E}(f(W(T)) \mid \mathcal{F}_t) = \varphi(t, W(t))$, for some function $\varphi = \varphi(t, x)$ that is given by an explicit formula involving an integral of f and the density of the normal distribution. (We encountered this in class, and again on your previous homework.) Show that $\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = 0$.

   (b) Show that $f(W(T)) = \boldsymbol{E}f(W(T)) + \int_0^T \partial_x \varphi(s, W(s))\,dW(s)$.

   [Now using an approximation argument one can show that for *any* $\mathcal{F}_T$ measurable random variable ξ, we must have $\xi = \boldsymbol{E}\xi + \int_0^T \sigma(s)\,dW(s)$ for some adapted process σ. This is called the *Itô representation theorem.*

   Using this, one can quickly show that if M is any (square integrable) martingale with respect to the Brownian filtration, then we must have $M(T) = \boldsymbol{E}M(0) + \int_0^T \sigma(s)\,dW(s)$. This is called the *martingale representation theorem.* (Note that Itô integrals are always martingales. The martingale representation theorem guarantees the converse.) This will (almost surely) come up in the second mini.]

Assignment 5 (assigned 2020-02-06, due 2020-02-13).

Unless otherwise noted, $\{\mathcal{F}_t\}$ is the Brownian filtration, and W is a standard Brownian motion. In all problems that ask you to simply compute something, you should also explain how you arrived at the answer and not simply state the answer.

Note that if X is a process with mean 0 independent increments (i.e. $X(t) - X(s)$ is independent of $\mathcal{F}_s^X$ *and* $\boldsymbol{E}(X(t) - X(s)) = 0$), then X must be a martingale with respect to the filtration generated by X. The converse is false. Here is a counter example.

1. Let $M(t) = \int_0^t W(s)\,dW(s)$.
   (a) For $s < t$, compute $\boldsymbol{E}\big((M(t) - M(s))^2 \bigm| \mathcal{F}_s\big)$.
   (b) Compute $\boldsymbol{E}(M(t) - M(s))^2 W(s)^2$ and $\boldsymbol{E}(M(t) - M(s))^2 \boldsymbol{E} W(s)^2$, and verify that they are not equal. Conclude $M(t) - M(s)$ is not independent of $\mathcal{F}_s$.
   (c) *(Unrelated)* Given $\lambda \in \mathbb{R}$ and $s < t$ show that
$$\boldsymbol{E}\big(e^{\lambda(M(t)-M(s))} \bigm| \mathcal{F}_s\big) = 1 + \frac{\lambda^2}{2} \int_s^t \boldsymbol{E}\big(e^{\lambda(M(r)-M(s))} W(r)^2 \bigm| \mathcal{F}_s\big)\,dr\,.$$

2. *(Itô and martingale representation theorems.)* Fix $T > 0$, and suppose $f = f(x)\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.
   (a) We know that $\boldsymbol{E}(f(W(T)) \mid \mathcal{F}_t) = \varphi(t, W(t))$, for some function $\varphi = \varphi(t,x)$ that is given by an explicit formula involving an integral of f and the density of the normal distribution. (We encountered this in class, and again on your previous homework.) Show that $\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = 0$.
   (b) Show that $f(W(T)) = \boldsymbol{E} f(W(T)) + \int_0^T \partial_x \varphi(s, W(s))\,dW(s)$.
   [Now using an approximation argument one can show that for *any* $\mathcal{F}_T$ measurable random variable ξ, we must have $\xi = \boldsymbol{E}\xi + \int_0^T \sigma(s)\,dW(s)$ for some adapted process σ. This is called the *Itô representation theorem*.
   Using this, one can quickly show that if M is any (square integrable) martingale with respect to the Brownian filtration, then we must have $M(T) = \boldsymbol{E}M(0) + \int_0^T \sigma(s)\,dW(s)$. This is called the *martingale representation theorem*. (Note that Itô integrals are always martingales. The martingale representation theorem guarantees the converse.) This will (almost surely) come up in the second mini.]

3. *(Leibniz' Rule).* Let $f(t,x)$ be a function of two variables, t and x, and assume that the partial derivatives $\partial_t f(t,x)$ and $\partial_x f(t,x)$ exist. If we replace x by a function $x(t)$ that is differentiable, then the total derivative of $f(t, x(t))$ is
$$\frac{d}{dt} f\big(t, x(t)\big) = \partial_t f\big(t, x(t)\big) + \partial_x f\big(t, x(t)\big) x'(t). \tag{6}$$
   In differential notation, we write this as
$$df\big(t, x(t)\big) = \partial_t f\big(t, x(t)\big)\,dt + \partial_x f\big(t, x(t)\big) x'(t)\,dt.$$
   Now let $g(s,x)$ be a function of two variables, s and x, and assume that the partial derivative $\partial_x g(s,x)$ exists. We can then define
$$f(t,x) = \int_0^t g(s,x)\,ds.$$
   The Fundamental Theorem of Calculus implies that the partial derivative of f with respect to t is
$$\partial_t f(t,x) = g(t,x).$$
   The partial derivative of f with respect to x is
$$\partial_x f(t,x) = \int_0^t \partial_x g(s,x)\,ds.$$
   Again we can replace x by a differentiable function $x(t)$. In this special case, (6) becomes
$$\frac{d}{dt}\Big(\int_0^t g\big(s, x(t)\big)\,ds\Big) = g\big(t, x(t)\big) + \Big(\int_0^t \partial_x g\big(s, x(t)\big)\,ds\Big)\,x'(t).$$
   This equation is called *Leibniz' Rule for Riemann integration.* In differential notation, we write Leibniz' Rule for Riemann integration as
$$d\Big(\int_0^t g\big(s, x(t)\big)\,ds\Big) = g\big(t, x(t)\big)\,dt + \Big(\int_0^t \partial_x g\big(s, x(t)\big)\,ds\Big)\,x'(t)\,dt. \tag{7}$$
   This rule says that when we are computing the differential with respect to t of $\int_0^t g(s, x(t))\,ds$, we must compute the differential with respect to both places t appears in this expression. According to the Fundamental Theorem of Calculus, the differential with respect to t in the upper limit of integration is $g(t, x(t))\,dt$. To that we must add the differential with respect to the t appearing as the argument of $x(t)$, and this requires that we differentiate with respect to x, obtaining $\partial_x g(s, x(t))$ under the integral sign, and then multiply this by the differential $x'(t)\,dt$ of $x(t)$.
   Under the same assumptions, namely that $g(s,x)$ is a function of two variables s and x and the partial derivative $\partial_x g(s,x)$ exists, and that $x(t)$ is a nonrandom differentiable function of t, *Leibniz' Rule for Itô integration* says that
$$\begin{aligned} d\Big(\int_0^t g\big(s, x(t)\big)\,dW(s)\Big) &= g\big(t, x(t)\big)\,dW(t) \\ &\quad + \Big(\int_0^t \partial_x g\big(s, x(t)\big)\,dW(s)\Big)\,x'(t)\,dt. \end{aligned} \tag{8}$$
   For $x \in \mathbb{R}$ and $t \geqslant 0$, we now define
$$I(t,x) = \int_0^t (x - s)\,dW(s).$$

(a) Use Leibniz' Rule for Itô integration (8) to compute the differential of $I(t,t)$.

(b) From the definition of $I(t,x)$, we have

$$I(t,x) = \int_0^t x\,dW(s) - \int_0^t s\,dW(s) = xW(t) - \int_0^t s\,dW(s),$$

and therefore

$$I(t,t) = \int_0^t t\,dW(s) - \int_0^t s\,dW(s) = tW(t) - \int_0^t s\,dW(s).$$

Compute the differential of $tW(t) - \int_0^t s\,dW(s)$ and check that it agrees with your answer in the first part.

(c) Is $I(t,t) = \int_0^t (t-s)\,dW(s)$ a martingale? Why or why not?

(d) What is $\boldsymbol{E}I(t,t)$?

4. This problem outlines how you would go about "solving" the Black-Scholes-Merton PDE. Suppose $c = c(t,x)$ solves $\partial_t c + rx\partial_x c + \frac{\sigma^2 x^2}{2}\partial_x^2 c = rc$, with boundary conditions $c(t,0) = 0$, linear growth as $x \to \infty$, and terminal condition $c(T,x) = (x-K)^+$.

(a) Set $y = \ln x$ and compute $\partial_x c$, $\partial_x^2 c$ in terms of y, $\partial_y c$ and $\partial_y^2 c$. Use this to find constants β_1, $\beta_2 \in \mathbb{R}$ such that $\partial_t c + \beta_1 \partial_y c + \beta_2 \partial_y^2 c = rc$.

(b) Let $\tau = T - t$, $z = y + \gamma_2 \tau$ and $v(\tau, z) = e^{\gamma_1 \tau} c(t,y)$. Find γ_1 and γ_2 so that $\partial_\tau v = \kappa \partial_z^2 v$ for some constant $\kappa > 0$. Express γ_1, γ_2 and κ in terms of σ^2 and r.

The equation you obtained for v above is called the *heat equation*, whose solution formula can be found in any standard PDE book. Namely, if we set $f(y) = v(0,y)$, then at times $\tau > 0$ the function v is given by

$$v(\tau, y) = \frac{1}{\sqrt{4\pi\kappa\tau}} \int_{\mathbb{R}} f(y-z) \exp\Big(\frac{-z^2}{4\kappa\tau}\Big)\,dz$$

(This is very similar to the formula you should have obtained in question 2.(a). In fact, by rescaling time one can derive the above formula using what you obtained in question 2.(a).)

(c) *(Optional)* Using the above formula for v, substitute back and derive the Black, Scholes, Merton formula for c. [While this is good practice, it is a little tedious. We will derive the formula in class using risk neutral measures.]

Assignment 6 (assigned 2020-02-13, due 2020-02-20).

1. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Let $\gamma > 0$ and consider a security that pays $S(T)^\gamma$ at time T. Compute the arbitrage free price of this security.

   HINT: Use the replicating portfolio argument to reduce this problem to finding the solution of a PDE with appropriate terminal and boundary conditions. Now look for a solution to these equations that is of the form $c(t,x) = f(t)g(x)$ for some functions f, g, and then find f and g explicitly.

2. *Question asked on a job interview (a few years ago)*

   > Determine the final value of a delta-hedge of a long call position if the realized volatility is different from the implied volatility.

   The question asked was the sentence above. Here is the same question posed in more detail. Let

   $$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

   be the price of a European call, expiring at time T with strike price K, if the stock price at time t is x, where

   $$d_\pm(T-t,x) = \frac{1}{\sigma_1\sqrt{T-t}}\Big[\log\frac{x}{K} + \Big(r \pm \frac{1}{2}\sigma_1^2\Big)(T-t)\Big].$$

   This call price formula assumes the underlying stock is a geometric Brownian motion with volatility $\sigma_1 > 0$. For this problem we take this to be the market price of the call. In other words, σ_1 is the *implied volatility*, the one that makes the Black-Scholes formula agree with the market price of the call.

   Suppose, however, that the underlying stock is really a geometric Brownian motion with volatility $\sigma_2 > 0$, i.e.,

   $$dS(t) = \alpha S(t)\,dt + \sigma_2 S(t)\,dW(t).$$

   We assume for most of this problem that σ_2 is constant. After we observe the stock price between times 0 and T, if we estimate the so-called *realized volatility*, we get σ_2. Consequently, the market price of the call at time zero is incorrect, although we do not know this at time zero.

   We set up a portfolio whose value at each time t we denote by $X(t)$. We begin with $X(0) = 0$. At each time t, the portfolio is long one European call and is short $\partial_x c(t, S(t)) = N(d_+(T-t, S(t)))$ shares of stock. This is the delta-hedge of the long call position.

   There is a cash position associated with this hedge which is often neglected. Here we keep track of it. We start with zero initial capital, and so at the initial time the portfolio has a cash position

   $$-c(0,S(0)) + S(0)\partial_x c(0,S(0)) = Ke^{-rT}N\big(d_-(T,S(0))\big),$$

   because we spend $c(0,S(0)) = S(0)N(d_+(T,S(0))) - Ke^{-rT}N(d_-(T,S(0)))$ to buy the call and we receive $S(0)\partial_x c(0,S(0)) = S(0)N(d_+(T,S(0)))$ when we short the stock. This cash is invested in a money market account with a constant continuously compounding interest rate r. At subsequent times, as we adjust the position in stock, we finance this by taking money from the money market account or depositing money into the money market account, depending on whether we are buying or selling stock, respectively. Therefore, the differential of the portfolio value is

   $$dX(t) = dc(t,S(t)) - \partial_x c(t,S(t))\,dS(t) + r\big[X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t))\big]\,dt$$

   for $0 \leqslant t \leqslant T$. The term $dc(t,S(t))$ accounts for the profit or loss from the long call position. The term $-\partial_x c(t,S(t))\,dS(t)$ accounts for the profit or loss from the short stock position. Finally, since $X(t)$ is the total portfolio value, if we take into account the long call and the short stock positions, we see that the cash position is

   $$X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t)).$$

   This is invested at interest rate r. The term

   $$r\Big[X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t))\Big]\,dt$$

   in the above formula for $dX(t)$ keeps track of these interest earnings.

   (a) Determine the value of $X(T)$. In particular, discuss the relationship among σ_1, σ_2 and the sign of $X(T)$. HINT: Compute $d(e^{-rt}X(t))$.

   (b) How would the analysis change if, instead of being constant, σ_2 is an adapted process $\sigma_2(t)$?

3. *(Asian options)* Let S be a geometric Brownian motion with mean return rate α and volatility σ, modelling the price of a stock. Let $Y(t) = \int_0^t S(s)\,ds$.

   (a) Let $f = f(t,x,y)$ be any function that is C^2 in x, y and C^1 in t. Find a condition on f such that $X(t) = f(t, S(t), Y(t))$ represents the wealth of an investor that has a portion of his wealth invested in the stock, and the rest in a money market account with return rate r.

   HINT: We know that if X represents the wealth of such an investor and $\Delta(t)$ is the number of shares of the stock held at time t, then $dX(t) = \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))\,dt$.

   Let $V = V(x,y)$ be a function and consider a derivative security that pays $V(S(T), Y(T))$ at time T. Note, if $V(x,y) = (y/T - K)^+$ then this is exactly an Asian option with strike price K.

   (b) Suppose $c = c(t,x,y)$ is a function such that $c(t, S(t), Y(t))$ is the arbitrage free price of this security at time t. Assuming c is C^1 in t and C^2 in x, y when $t < T$, find a PDE and boundary conditions satisfied by c.

   [The PDE you obtain will be similar to the Black-Scholes PDE, but will also involve derivatives with respect to the new variable y. Unlike the case of European options, the PDE you obtain here will not have an explicit solution.]

   (c) Conversely, if c is the solution to the PDE you found in the previous part then show that the arbitrage free price of this security is exactly $c(t, S(t), Y(t))$.

Assignment 7 (assigned 2020-02-20, due 2020-02-27).

In light of your FINAL, solutions to this homework will post on 2020-02-27, and late homework will not be accepted.

1. The main idea behind arbitrage free pricing is to reproduce the pay-off of a derivative security by trading the underlying risky asset and a riskless money market account. At time t, let $S(t)$ be the price of the risky asset, $M(t) = e^{rt}$ the price of one share in the money market (that is assumed to have a constant return rate r), $X(t)$ be the value of a portfolio that holds $\Delta(t)$ shares of the risky asset, $\Gamma(t)$ shares of the money market. Then we should have

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t)\,.$$

Assuming that no external cash is injected into the portfolio we should also have

$$dX(t) = \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))\,dt\,.$$

Use these two equations to derive the *self-financing* condition

$$S(t)\,d\Delta(t) + d[S,\Delta](t) + M(t)\,d\Gamma + d[M,\Gamma](t) = 0\,.$$

2. Let W and B be two independent (one dimensional) Brownian motions. Let M, N be defined by

$$M(t) = \int_0^t W(s)\,dB(s) \qquad \text{and} \qquad N(t) = \int_0^t B(s)\,dW(s)\,.$$

Show $[M,N] = 0$. Also verify $\boldsymbol{E}M(t)^2\boldsymbol{E}N(t)^2 \neq \boldsymbol{E}M(t)^2N(t)^2$, and show that M, N are not independent even though $[M,N] = 0$.

3. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $K, T > 0$ and consider a derivative security that pays $(S(T)^2 - K)^+$ at maturity T. Compute the arbitrage free price of this security at any time $t \in [0,T)$. Your answer may involve r, σ, K, t, T, S, and the CDF of the normal distribution, but not any integrals or expectations.

   HINT: The simplest way to solve this problem is to use the risk neutral pricing formula, along with the explicit Black-Scholes formula you already know.

4. Let W be a d-dimensional Brownian motion with an invertible covariance matrix A. (This means that W is a continuous d-dimensional process such that for $s < t$, $W(t) - W(s) \sim N(0,(t-s)A)$, and is independent of $\mathcal{F}_s$.) Let b be a bounded adapted process, and suppose

$$dX(t) = b(t)\,dt + dW(t)\,.$$

Let $T > 0$. Find a measure $\tilde{P}$ such that $\tilde{P}$ and P are equivalent, and X is a martingale under $\tilde{P}$, up to time T. Express $d\tilde{P}$ as $Z(T)\,dP$ for some process Z you find explicitly. [HINT: Any covariance matrix A can be expressed in the form $\sigma\sigma^*$, where σ is a $d \times d$ matrix and σ^* is the transpose of σ.)]