46-944 Stochastic Calculus for Finance I: Final.

2020-03-04

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck* $\ddot{\smile}$.

Unless otherwise stated, W denotes a standard (one dimensional) Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

$\boxed{5}$ 1. Let $0 < r < s < t$. Compute $\boldsymbol{E}\big(W(s)W(t) \bigm| \mathcal{F}_r\big)$. Express your answer in terms of r, s, t and $W(r)$ without involving expectations or integrals.

$\boxed{5}$ 2. Let $W = (W_1, W_2)$ be a standard two dimensional Brownian motion. Find two adapted processes b and σ such that

$$X(t) = e^{-2t} W_1(t) + \int_0^t \sigma(s)\, dW_2(s) + \int_0^t b(s)\, ds$$

is a Brownian motion.

$\boxed{5}$ 3. Let $\sigma, K, T > 0$, $\alpha \in \mathbb{R}$, and $r \geqslant 0$ be given constants. Consider a financial market consisting of a stock and a money market account. The money market account has continuously compounded interest rate r. The stock price, denoted by S, is a geometric Brownian motion with mean return rate α and volatility σ. Let $\Delta(t)$ denote the (possibly fractional) number of shares held at time t by the replicating portfolio of a European call option with strike K and maturity T. Let $C(t)$ denote the wealth at time t of this replicating portfolio held in the money market account.

(a) Write down formulae for $\Delta(t)$ and $C(t)$ in terms of t and $S(t)$, and the model parameters σ, K, etc. No derivation / justification is required. Your formula may use the CDF of the standard normal, but must not involve any expectations or integrals.

(b) If $S(T) > K$, compute $\Delta(T)$ and $C(T)$. Your answer may involve the model parameters (r, K, T, etc.), but must not involve expectations, integrals or even the CDF of the normal.

(c) If $S(T) < K$, compute $\Delta(T)$ and $C(T)$. Your answer may involve the model parameters (r, K, T, etc.), but must not involve expectations, integrals or even the CDF of the normal.

$\boxed{5}$ 4. Compute $\boldsymbol{E}\Big[W(t)^2 \displaystyle\int_0^t W(s)\, dW(s)\Big]$. Express your answer in terms of t without involving W, expectations or integrals.

$\boxed{5}$ 5. Let $\alpha \in \mathbb{R}$, $\sigma, T > 0$ and S be a geometric Brownian motion with mean return rate α and volatility σ.

(a) Find an equivalent measure $\tilde{\boldsymbol{P}}$ such that up to time T, the process S is a martingale under $\tilde{\boldsymbol{P}}$. Express the measure $\tilde{\boldsymbol{P}}$ in the form $d\tilde{\boldsymbol{P}} = Z(T)d\boldsymbol{P}$ for a process Z that you find explicitly.

(b) If $0 < s < t < T$, compute $\tilde{\boldsymbol{E}}[S(s)S(t)]$, where $\tilde{\boldsymbol{E}}$ denotes expectations with respect to the measure $\tilde{\boldsymbol{P}}$. Express your answer in terms of r, s, t, T without using S, expectations or probabilities.

$\boxed{5}$ 6. Let $M(t) = \displaystyle\int_0^t e^{-s} dW(s)$. Find a non-random function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(1) \neq 0$ and the process N defined by $N(t) = f(e^t M(t))$ is a martingale. Your final answer should write down a formula for $f(x)$ for every $x \in \mathbb{R}$. (Your formula may involve unsimplified Riemann integrals.)

Here $f = f(x)$ is a function of one real variable x. To avoid getting confused I recommend you define a new function $g(t, x) = f(e^t x)$, and observe $N(t) = g(t, M(t))$.

$\boxed{5}$ 7. Let $\sigma_1 > 0$, $\sigma_2 = 10\sigma_1$, $0 < T_1 < T$, α_1, $\alpha_2 \in \mathbb{R}$ and $r \geqslant 0$ be given constants, and consider a financial market consisting of a stock and a money market account. The money market account has continuously compounded interest rate r. We expect that at time T_1 a global pandemic will decimate the population, leading to a stock market crash. Thus we model the stock price, denoted by S, by a geometric Brownian motion with mean return rate α_1 and volatility σ_1 up to time T_1. After time T_1, we assume that the stock price will follow a geometric Brownian motion with mean return rate α_2 and volatility $\sigma_2 = 10\sigma_1$. Explicitly, this means

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)dW(t)\,, \qquad \text{where } \alpha(t) = \begin{cases} \alpha_1 & t \leqslant T_1\,, \\ \alpha_2 & t > T_1\,, \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} \sigma_1 & t \leqslant T_1\,, \\ \sigma_2 & t > T_1\,. \end{cases}$$

Consider a European call option on S with strike $K > 0$ and maturity $T > T_1$. Find the arbitrage free price of this option at time a time $t \in [0, T_1]$. You may leave your answer as an unsimplified Riemann integral, as long as it only involves $t, S(t)$, the model parameters (α_1, α_2, σ_1, etc.) and functions you have explicitly found formulae for. Your answer must not use probabilities or expectations.

HINT: Express $S(T)$ in terms of $S(T_1)$, and express $S(T_1)$ in terms of $S(t)$.
Disclaimer: This problem is entirely hypothetical, and any similarity to current events is purely coincidental.

$\boxed{5}$ 8. Let B and W be two independent, standard, one dimensional Brownian motions. Compute $\boldsymbol{E} \displaystyle\int_0^{B(t)^2} W(s)^2\,ds$. Express your answer as a function of t without involving W, B, integrals, expectations or probabilities.