CHAPTER 1

Introduction

The price of a stock is not a smooth function of time, and standard calculus tools can not be used to effectively model it. A commonly used technique is to model the price, S, as a *geometric Brownian motion*, given by the *stochastic differential equation (SDE)*

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)\,,$$

where α and σ are parameters, and W is a Brownian motion. If $\sigma = 0$, this is simply the ordinary differential equation

$$dS(t) = \alpha S(t)\,dt \quad \text{or} \quad \partial_t S = \alpha S(t)\,.$$

This is the price assuming it grows at a rate α. The $\sigma\,dW$ term models *noisy fluctuations* and the first goal of this course is to understand what this means. The mathematical tools required for this are Brownian motion, and Itô integrals, which we will develop and study.

An important point to note is that the above model can not be used to *predict* the price of S, because randomness is built into the model. Instead, we will use this model to *price securities*. Consider a *European call option* for a stock S with strike prices K and maturity T (i.e. this is the right to buy the asset S at price K at time T). Given the stock price $S(t)$ at some time $t \leqslant T$, what is a fair price for this option?

Seminal work of Black and Scholes computes the fair price of this option in terms of the time to maturity $T - t$, the stock price $S(t)$, the strike price K, the model parameters α, σ and the interest rate r. For notational convenience we suppress the explicit dependence on K, α, σ and let $c(t, x)$ represent the price of the option at time t given that the stock price is x. Clearly $c(T, x) = (x - K)^+$. For $t \leqslant T$, the Black-Scholes formula gives

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x))$$

where

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,.$$

Here r is the interest rate at which you can borrow or lend money, and N is the CDF of a standard normal random variable. (It might come as a surprise to you that the formula above is independent of α, the mean return rate of the stock.)

The second goal of this course is to understand the derivation of this formula. The main idea is to find a *replicating strategy*. If you've sold the above option, you hedge your bets by investing the money received in the underlying asset, and in an interest bearing account. Let $X(t)$ be the value of your portfolio at time t, of which you have $\Delta(t)$ invested in the stock, and $X(t) - \Delta(t)$ in the interest bearing account. If we are able to choose $X(0)$ and Δ in a way that would guarantee $X(T) = (S(T) - K)^+$ almost surely, then $X(0)$ must be the fair price of this option. In order to find this strategy we will need to understand SDEs and the Itô formula, which we will develop subsequently.

The final goal of this course is to understand *risk neutral measures*, and use them to provide an elegant derivation of the Black-Scholes formula. If time permits, we will also study the *fundamental theorems of asset pricing*, which roughly state:

(1) The *existence* of a risk neutral measure is equivalent to the market having no arbitrage (i.e. you can't make money without taking risk).
(2) *Uniqueness* of a risk neutral measure is equivalent to the market having no arbitrage, and that all derivative securities can be hedged (i.e. for every derivative security we can find a replicating portfolio).