21-268 Multidimensional Calculus: Midterm 2.

2020-03-25

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not*, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *Unless you have made prior arrangements, you must join the regular class Zoom video call* with video enabled *for the duration of the exam. Please use the chat to message me privately if you have questions. Do NOT use the mic.*
- *You must scan and turn in your exam via Gradescope. Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *If a question you are asked, or a result you want to use, has been done in class / the homework, you do* not *have to re-derive it. Simply reference it appropriately, quote/use it as needed.*
- *You have 50 minutes. The exam has a total of 4 questions and 20 points.*
- *The questions are roughly ordered by difficulty. Good luck* ☺.

$\boxed{4}$ 1. Let $f\colon \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = 2|x|^2 - x_1 x_2 + -4x_2 x_3 x_4$. For each of the points below, determine whether f attains a local minimum, local maximum, local saddle, or none of the above.

(a) The point $a = (-1, 2, 1, 2)$.

(b) The point $a = (0, 0, 0, 0)$.

$\boxed{5}$ 2. Suppose $n, d \geqslant 1$, $f\colon \mathbb{R}^{n+d} \to \mathbb{R}$ is C^2, and $U \subseteq R^{n+d}$ is an n-dimensional subspace. Suppose further the function f *when restricted to* U attains a local minimum at a point a. (That is, there exists $\varepsilon > 0$ such that if $|u - a| < \varepsilon$ and $u \in U$, then $f(a) \leqslant f(u)$.) True or false: For every $u \in U$, we must have $u \cdot (Hf_a u) \geqslant 0$. Prove it, or find a counter example. [Note, here U is an n-dimensional *subspace*.]

$\boxed{5}$ 3. Let S be the surface $2xy + z^2 = -1$ in $\mathbb{R}^3$, and $a = (-1, 1, 1) \in S$. Let U be the tangent space of S at a. Do there exist two non-zero vectors u, v such that $u \in U$, $v \in \mathbb{R}^3$, and $u \cdot v = \frac{1}{2}|u||v|$? If yes, find one pair. If no, prove it.

$\boxed{6}$ 4. Let $a = (0, 2, 1) \in \mathbb{R}^3$, $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a C^2 function such that

$$f(a) = 0 = (0,0) \in \mathbb{R}^2, \qquad Df_a = \begin{pmatrix} 2 & 6 & 8 \\ 2 & 6 & 9 \end{pmatrix}, \qquad (Hf_1)_a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \qquad (Hf_2)_a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Here $f = (f_1, f_2)$, and $(Hf_1)_a$ and $(Hf_2)_a$ are the Hessian's of the coordinate functions f_1 and f_2 respectively, at the point a. For some small $\varepsilon > 0$, one of the following functions necessarily exists. Decide which one. No justification required.

(i) A C^2 function $g\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that $g(0) = (0, 1)$ and $f(g_1(t), 2 + t, g_2(t)) = 0$ for all $t \in (-\varepsilon, \varepsilon)$.

(ii) A C^2 function $g\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that $g(0) = (0, 2)$ and $f(g_1(t), g_2(t), 1 + t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$.

(iii) A C^2 function $g\colon B((0, 1), \varepsilon) \to \mathbb{R}$ such that $g(0, 1) = 2$ and $f(s, g(s, t), t) = 0$ for all $(s, t) \in B((0, 1), \varepsilon)$.

If you decided one of the first two functions above exists, then compute $g_1''(0)$. If you decided the third function above exists, then compute $\partial_1 \partial_2 g(0, 1)$. (Partial credit for computing $g_1'(0)$ in the first case, or $\partial_1 g(0, 1)$ in the second case.)