$U \subseteq \mathbb{R}^3 \longrightarrow U \supseteq \{x \mid x_3 > 0,\ |x| < 2,\ x_1^2 + x_2^2 < 1\}$

2017 final Q8

find vol of U

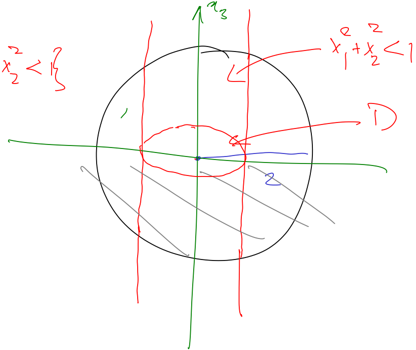
$D = \{x_3 = 0 \ \& \ x_1^2 + x_2^2 < 1\}$

$$\mathrm{vol}(U) = \int_U 1 \, dV = $$

Fubini

$$\iint_{x_1\ x_2} \left(\int_{x_3 = 0}^{\sqrt{4 - x_1^2 - x_2^2}} 1 \, dx_3 \right) dx_1 \, dx_2$$

$\in$ Disk D



$$= \int_{(x_1,x_2)\in D} \left(\int_{x_2=(\ )}^{(\ )} 1 \, dx_3 \right) dA = \int_{(x_1,x_2)\in D} \sqrt{4-x_1^2-x_2^2} \, dA$$

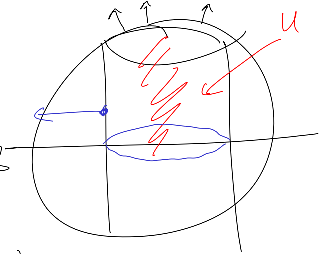
= Polar coordinates & solve.

Another method: $\mathrm{vol}(U) = \int_U 1 \, dV$

Suppose we find a fn $F : \mathbb{R}^3 \to \mathbb{R}^3$ $\ni$ $\nabla \cdot F = 1$

Then $\mathrm{vol}(U) = \int_U (\nabla \cdot F) \, dV \overset{\text{div Thm}}{=} \int_{\partial U} F \cdot \hat{n} \, dS$ & evaluate.

Chose $F = \cancel{\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$



$$\Rightarrow \int_{\partial U} F \cdot \hat{n}\, dS = \underbrace{\int F \cdot \hat{n}\, dS}_{\text{Top curved cap}} + \underbrace{\int F \cdot \hat{n}\, dS}_{\text{Bot. disk}}$$

$F = 0$

0

surface

$U \longrightarrow$ Hole (open set)

$\Sigma \rightarrow \partial U \longrightarrow$ bdry of U = bot disk + curved cyl + top sph cap

Q: $\partial \Sigma$ = bdry of $\Sigma = \emptyset$

Eg : Compute $\underbrace{\int_{\text{Top sph cap}}}_{\Sigma} F \cdot \hat{n} \, dS$

$\Sigma = \{ x \in \mathbb{R}^3 \mid |x| = 2 \;\&\; x_1^2 + x_2^2 < 1 \;\&\; x_3 > 0 \}$

$F = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$.

① Param the surf: $\varphi(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ \sqrt{4 - x_1^2 - x_2^2} \end{pmatrix}$

② $\int_\Sigma F \cdot \hat{n} \, dS = \int_D F \circ \varphi \;\; \partial_1 \varphi \times \partial_2 \varphi \;\; dA$

① Param Σ by $\varphi(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{4-r^2} \end{pmatrix}$

② $\int_\Sigma F \cdot \hat{n}\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \begin{pmatrix} 0 \\ 0 \\ \sqrt{4-r^2} \end{pmatrix} \cdot (\partial_r \varphi \times \partial_\theta \varphi)\, dr\, d\theta$

$F = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$

Alternate try: $\int_\Sigma \nabla \times G \cdot \hat{n}\, dS = \oint_{\partial\Sigma} G \cdot d\ell$

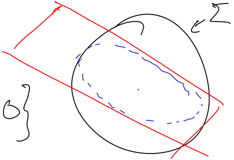
$\uparrow$ $\nabla \times G$

Find G + $\nabla \times G = F$. $\boxed{\text{Claim } \nexists\, G \text{ + } \nabla \times G = F = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}}$

Know $\nabla \cdot (\nabla \times G) = 0$
But $\nabla \cdot F = 1$
$\Rightarrow \nexists G \ni \nabla \times G = F$.

HW 15] $F = \begin{pmatrix} 2x \\ y^2 \\ z^2 \end{pmatrix}$, $\Sigma \subseteq \mathbb{R}^3 = \{x \mid |x| < 1\}$

$\Gamma = \Sigma \cap \{x + 2y + 3z = 0\}$



Find $\int_\Gamma F \cdot dl = \int_\Sigma (\nabla \times F) \cdot \hat{n}\, dS$

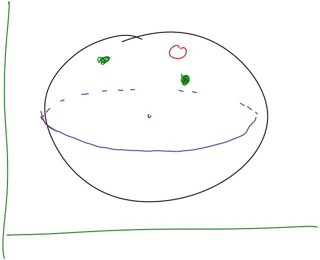
$\partial \Sigma = \Gamma$

$$\underset{\text{curl}}{\nabla \times F} = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$F = \underset{\text{grad}}{\nabla \varphi}$, $\varphi = x^2 + \frac{y^3}{3} + \frac{z^3}{3}$

$$\partial_y \left(\int_0^1 \int_0^1 f(x, y, z) \, dx \, dz \right) = \int_0^1 \int_0^1 \partial_y f(x, y) \, dx \, dz$$

$$\partial_y \int_0^y f(x, s) \, ds$$



$$\frac{d}{dt} \left(\underbrace{\int_0^t e^{-ts^2} \, ds} \right)$$

$\lim_{x \to a} f(x)$?

$a \in \mathbb{R}^d$

(1) $f \to$ ets ✓

Sin Cos exp

$\frac{\text{poly}}{\text{poly}}$

$\boxed{\text{with no 0's in denom}}$

(1D) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

multi d $\lim_{h \to 0} \frac{f(x+h) - f(x) - Df_x h}{|h|}$ ← 0 in denom

$$\lim_{(x,y) \to 0} \frac{x^a y^b}{|x|^c + |y|^d}$$

deg = a+b

c and d

$$\lim_{(x,y,z)\to 0} \frac{xyz}{x^2+y^2+z^4}$$

$z = t$

Try (1) $x = t$
$y = at$
$z = bt$

$$\lim_{t\to 0} \frac{ab\,t^3}{t^2+a^2t^2+b^4t^4} \to 0$$

Try (2): $x = \alpha t^a$, $y = \beta t^b$, $z = \gamma t^c$

$$f(x,y,z) = \frac{t^{a+b+c}}{t^{2a}+t^{2b}+t^{4c}}$$

$\longleftarrow$ deg $(a+b+c)$

$\longleftarrow$ min $(2a, 2b, 4c)$

Q: Can you make $a+b+c \le \min(2a, 2b, 4c)$ $(a,b,c>0)$

$$\left|\frac{xyz^2}{x^2+y^2+z^4}\right| = \left(\frac{x^2}{x^2+y^2+z^4}\right)^{1/2}\left(\frac{y^2}{x^2+y^2+z^4}\right)^{1/2}\left(\frac{z^4}{x^2+y^2+z^4}\right)^{1/4}(x^2+y^2+z^4)^{1/4}$$

$$\le (x^2+y^2+z^4)^{1/4} \longrightarrow 0$$

Want $\le \varepsilon$

$$\underbrace{(\delta^2+\delta^2+\delta^4)^{1/4}}_{\le \delta^{1/2}(2+\delta^2)}$$

$(\delta<1)$

$\leq 3\,\delta^{1/2}$ & choose $\delta \leq \boxed{\dfrac{\varepsilon^2}{9}}$

$f, \underbrace{g_1, \dots g_n}: \mathbb{R}^{\boxed{d+n}} \longrightarrow \mathbb{R}$

f constr $\{g = c\}$ (or $g_1 = c_1$ & $g_2 = c_2$ ... $g_n = c_n\}$)

is an n-dim manifold.

$Df = \sum \lambda_i Dg_i \qquad (Df, Dg_i \in \mathbb{R}^{n+d})$

HW 9 Q4:

Open box: Surface area $3a^2$, max volume.

f ~~then~~ $V = xyz$ ($x, y, z \to$ side lengths).

$g = S = xy + 2(yz + xz)$ Consider $S = 3a^2$ ($g = 3a^2$).

Lag Mult: $\nabla f = \lambda \nabla g$

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} y + 2z \\ x + 2z \\ 2x + 2y \end{pmatrix} \quad \& \quad xy + 2(yz + xz) = 3a^2$$

$$z(y-x) = \lambda(y-x) \implies x = y \quad \text{or} \quad z = \lambda$$