21-268 Review Session

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Compute the derivative:

$$Df_{(x,y,z)} = \begin{bmatrix} 2x & 2y & -3z^2 \\ ye^z \cos(xy) & xe^z \cos(xy) & e^z \sin(xy) \end{bmatrix}$$

$$Df_{(0,1,1)} = \begin{bmatrix} 0 & 2 & -3 \\ e & 0 & 0 \end{bmatrix}$$

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Thus, the tangent line at $(0, 1, 1)$ is $\{(0, 1, 1) + t(0, 3, 2) : t \in \mathbb{R}\}$.

Question 2 (Leibniz's rule)

Suppose $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is C^1 and $f(x, y)$ is non-decreasing w.r.t y.
Show that $\partial_y \int_a^b f(x, y)\mathrm{d}x = \int_a^b \partial_y f(x, y)\mathrm{d}x$.
(Hint: Fubini's theorem and FTC: $\partial_t \int_c^t g(s)\,\mathrm{d}s = g(t)$.)

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Solution: By FTC,

$$\partial_y \int_a^b f(x, y) \, \mathrm{d}x = \partial_y \int_a^b f(x, y) - f(x, 0) \, \mathrm{d}x = \partial_y \int_a^b \int_0^y \partial_t f(x, t) \, \mathrm{d}t \, \mathrm{d}x$$

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By Fubini's theorem ($\partial_t f(x, t) \geq 0$),

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By the extreme value theorem, since $\partial_t f(x, t)$ is continuous on $[a, b] \times [0, y]$ (compact), we know that $\exists M \geq 0$ such that $\forall (x, t) \in [a, b] \times [0, y]$, $|\partial_t f(x, t)| \leq M$.

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So, Fubini's theorem still applies.

Question 3

Let $f : [0,1]^2 \to \mathbb{R}^3$ be defined as $f(x,y) = xy$. Let $G \subset \mathbb{R}^3$ be the graph of f. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $F(x,y,z) = (\frac{1}{2}y^2, xy, xy)$. Compute $\oint_{\partial G} F \cdot \mathrm{d}\ell$, where ∂G is traversed counterclockwise w.r.t the upward pointing normal vector.

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Solution: We first parametrize G using $\varphi : [0,1]^2 \to G$ defined as $\varphi(u,v) = (u, v, uv)$. We can compute the unit normal of G:

$$\hat{n} = \frac{\partial_u \varphi \times \partial_v \varphi}{|\partial_u \varphi \times \partial_v \varphi|} = \frac{1}{\sqrt{u^2+v^2+1}}(-v, -u, 1)$$

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By Stoke's theorem,

$$\begin{aligned}\oint_{\partial G} F \cdot \mathrm{d}\ell &= \int_G \nabla \times F \cdot \hat{n}\, \mathrm{d}S = \int_G (x, -y, 0) \cdot \hat{n}\, \mathrm{d}S \\ &= \int_{[0,1]^2} (u, -v, 0) \cdot \frac{1}{\sqrt{u^2+v^2+1}}(-v, -u, 1)\, \mathrm{d}A \\ &= \int_{[0,1]^2} 0\, \mathrm{d}A = 0\end{aligned}$$

Question 4

Assuming you are allowed to use the mean value theorem in 1d, prove the mean value theorem in $\mathbb{R}^n$:
Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. For any $a, b \in \mathbb{R}^n$, there exists $\theta \in (0, 1)$ such that

$$f(b) - f(a) = (b - a) \cdot \nabla f((1 - \theta)a + \theta b)$$

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Solution
Define a function $g : [0, 1] \to \mathbb{R}$ by $g(t) = f((1 - t)a + tb)$. g is differentiable and

$$g'(t) = \nabla f((1 - t)a + tb))(b - a)^T = (b - a) \cdot \nabla f((1 - t)a + tb))$$

By mean value theorem for 1d,

$$f(b) - f(a) = g(1) - g(0) = g'(\theta) = (b - a) \cdot \nabla f((1 - \theta)a + \theta b)$$

for some $\theta \in (0, 1)$

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Solution: We differentiate the whole equation with respect to x, then

$$2xf(x) + x^2 f'(x) + f'(x)e^{f(x)} = 1$$

So

$$x^2 f'(x) + f'(x)e^{f(x)} = 1 - 2xf(x)$$

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Remark. A more complicated version of this problem would write $x^2 y + e^y = x$ and ask you when can you write one variable as a function of the other (locally), and compute the derivative. You need to use implicit function theorem.

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We can prove it by optimizing $f(x, y) = \sqrt{xy}$ subject to $g(x, y) = \frac{x+y}{2} = c$ for some $c > 0$.

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For $x, y \neq 0$, $\nabla f(x, y) = (\frac{\sqrt{y}}{2\sqrt{x}}, \frac{\sqrt{x}}{2\sqrt{y}})^T, \nabla g(x, y) = (\frac{1}{2}, \frac{1}{2})^T$

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We want $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda \in \mathbb{R}$. Then $\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{y}}$, which implies $x = y = c$. $f(c, c) = c$.

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We also need to check the boundary points $(0, 2c)$ and $(2c, 0)$. $f(0, 2c) = f(2c, 0) = 0$. So the constrained maximum of f is c, which implies $f(x,y) \leq g(x,y)$ when $x, y \geq 0$.

Things you need to know for the final

Definitions:

- Open sets and closed sets in $\mathbb{R}^d$
- ε-δ definition of limits
- Continuity of functions
- Directional and partial derivatives
- Differentiability of functions
- Curve, surface, and manifold
- Tangent planes and tangent spaces
- Parametric curves
- Higher order derivatives
- Riemann integrals (double and triple integrals)
- Line integrals, arc length integrals, and surface integrals
- Conservative and potential forces

Theorems you should know

Theorems:

- Algebra of limits and continuous functions
- Differentiability $\Rightarrow$ Continuity & Existence of all directional derivatives
- All partial derivatives exist & are continuous $\Rightarrow$ Differentiability
- Chain rule
- Necessary and sufficient conditions for local maxima/minima
- Sylvester's law of signs
- Mean value theorem
- Taylor's theorem
- Inverse and implicit function theorem.
- Tangent space of $\{f(x) = c\}$
- Constrained optimization/Lagrange multiplier
- Fubini's theorem
- Change of variable formula
- Fundamental theorem of line integral
- Invariance of parametrizations
- Greens, Stokes, Divergence theorem

Good Luck for the Final!!!