Tgt spaces: (1) Γ = Graph of a fn $= \{(x,y) \mid x \in \mathbb{R}^d, y \in \mathbb{R}^n \text{ \& } y = f(x)\}$

$(f : \mathbb{R}^d \to \mathbb{R}^n, C^1)$

Tgt space at $\underline{a} = (b, f(b)) = \{(u,v) \mid u \in \mathbb{R}^d, v \in \mathbb{R}^n, v = Df_b u\}$. $\mathbb{R}^{d+n}$

(2) $g : \mathbb{R}^{n+d} \to \mathbb{R}^n$, C^1, $c \in \mathbb{R}^n$, $M = \{g = c\} = \{z \in \mathbb{R}^{n+d} \mid g(z) = 0\}$.

$a \in M$ $(g(a) = c)$. Tgt space at $a = \ker(Dg_a)$

Consistency check: ((2) $\Rightarrow$ (1)).

Let $g(x,y) = y - f(x)$. Γ = Graph of f.

$M = \{g = 0\} = \Gamma$

(1) $\Rightarrow$ Tgt space of Γ at $a = (b, f(b)) = \{(u,v) \mid v = Df_b\, u\}$

(2) $\Rightarrow$ Tgt spce of M at $a = (b, f(b)) = \ker\left(Dg_{(b,f(b))}\right)$

$\qquad\qquad \overset{\shortparallel}{\Gamma}$

$$= \ker\left(\underbrace{\begin{bmatrix} -Df_b & I \end{bmatrix}}_{n+d} \right) \Big\} n$$

$n \times d \qquad\qquad n \times n$

Note $Dg_{(b,f(b))} \begin{pmatrix} u \\ v \end{pmatrix} = -Df_b\, u + v \qquad (u \in \mathbb{R}^d,\ v \in \mathbb{R}^n)$

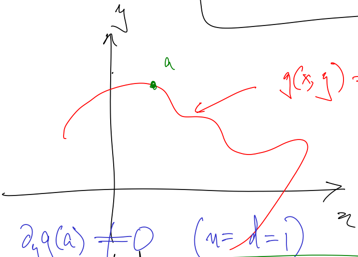
$$\begin{pmatrix} u \\ v \end{pmatrix} \in \ker\left(Dg_{(b, f(b))} \right) \iff -Df_b\, u + v = 0 \iff \boxed{v = Df_b\, u}$$

Same as (1)

How do u check (1) $\Rightarrow$ (2) :

$$g(x, y) = c$$

Near a, solve $g(x,y) = c$ for y & write $y = h(x)$



$\partial_y g(a) \neq 0$ $(n = d = 1)$

Impl fn thm : This is possible if the sub matrix of Dg_a with all rows & last n columns is inv

M near a is the graph of the fn h

$\Rightarrow$ Tgt space of $M = \{ (u, v) \mid v = Dh_b\, u \}$

Write Dh in terms of Dg & verify

$$v = Dh_b\, u \iff \begin{pmatrix} u \\ v \end{pmatrix} \in \ker(Dg)$$

(Check this by using $g(x, h(x)) = c$ & diff both sides)

Dim count

$g : \mathbb{R}^{n+d} \to \mathbb{R}^n$

$g(z) = c \in \mathbb{R}^n$

$\iff$ $g_1(z) = c_1$, $g_2(z) = c_2$, $\vdots$, $g_n(z) = c_n$

$n+d$ vars

n - eqns.

solve for $\boxed{n}$ vars.

Impl thm ① $g : \mathbb{R}^{d+1} \to \mathbb{R}^1$, $\partial_y g(a) \neq 0$ $\quad g(x, \underset{\underset{\mathbb{R}}{\cap}}{y}) = c$

② $g : \mathbb{R}^{d+n} \to \mathbb{R}^n$ $\quad$ sub matrix of Dg_a with all rows & last n cols is inv $\quad g(x,y) = c$

③ $g : \mathbb{R}^{d+n} \to \mathbb{R}^{n}$

& Rank $Dg_a = n$ $\quad (\Leftarrow)$ $\exists\, n$ colums + sub matrix of Dg with these n cols & all rows is inv)

& can solve for the n coordinates corr to those cols.

$$\int_{\mathbb{R}^3} f \, dV = \lim_{R \to \infty} \int_{B(0,R)} f \, dV$$

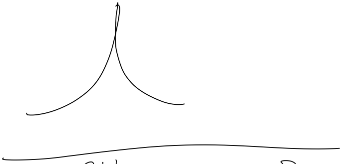
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \to \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx$$

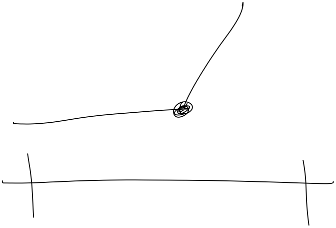
$$\neq$$

$$\lim_{R \to \infty} \int_{-R}^{2R} f(x) \, dx$$

① f cts ~~bdd~~ on C closed bdd $\Rightarrow$ ~~int~~ f is int over C

② disc of f are "small" $\longrightarrow$





$$g : \mathbb{R}^{2+1} \longrightarrow \mathbb{R}^2$$

Eg 1 $\quad Dg = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

$\longrightarrow$ not inv.

$\longrightarrow$ inv

$\longrightarrow$ inv.

Impl fn thm $\Rightarrow$ x_1 & x_3 can be solved for & expressed as C^1 fns of x_2

Eg 2: $g: \mathbb{R}^{2+1} \to \mathbb{R}$ $Dg = \begin{pmatrix} 1 & 0 & 3 \end{pmatrix}$

$\boxed{} \to$ inv

$\boxed{} \to$ inv $\to$ impl fn thm $\Rightarrow x_1$ can be expressed as a C^1 fn of x_2 & x_3

$\boxed{} \to$ not inv

---

(1) $T: \mathbb{R} \to \mathbb{R}^3$

Solve $Tx = y_0$ for some given y_0

(most of the time $\to$ impossible)

Exact image of T is a 1D subspace

(2) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Solve $Tx = y_0$ for some given y_0

(most of the time possible)

$\operatorname{Img}(T) = \mathbb{R}^3$

(3) $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

Solve $Tx = y_0 \longleftarrow$ 1d subspace of solutions

Change variable (Polar)

$dx\,dy \longrightarrow r\,dr\,d\theta \xrightarrow{\text{eval}}$

$$\int_{x,y} f(x,y)\,\cancel{dx\,dy} \xrightarrow{\text{Polar}} \int_{r,\theta} f(r,\theta)\, r\, \cancel{dr\,d\theta}$$

dA $\qquad$ dA

region in $(0,\infty)\times(0,2\pi)$

Fubini to evaluate

$$\int_{Cyl} f(x,y,z)\,dV \overset{\text{Fubini}}{=} \int_{z=0}^{1}\Big(\underbrace{\int_{x,y} dA}_{x-y \text{ coord change.}}\Big) = \int_{z=0}^{1}\Big(\int_{r,\theta} \cdots\, r\, dA\Big)$$

$$F = \begin{pmatrix} -y/2 \\ x/2 \end{pmatrix} \qquad \partial_1 F_2 - \partial_2 F_1 = 1$$

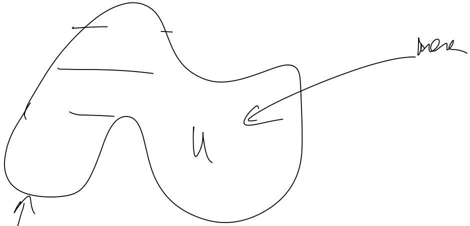
$$F = \begin{pmatrix} -y \\ 0 \end{pmatrix} \qquad \partial_1 F_2 - \partial_2 F_1 = 1$$

$$F = \begin{pmatrix} 0 \\ x \end{pmatrix} \qquad \text{"} \qquad = 1$$

$$\boxed{\partial_1 \varphi \times \partial_2 \varphi}$$

Rank $D\varphi = 2 \iff \partial_1 \varphi \times \partial_2 \varphi \neq 0$

$$D\varphi = \begin{pmatrix} \partial_1 \varphi & \partial_2 \varphi \end{pmatrix}$$



$\Gamma \longrightarrow$ Given a param of Γ

Find area encl by Γ

$$A = \int_U 1 \, dA = \int (\partial_1 F_2 - \partial_2 F_1) \, dA \quad \text{where} \quad F = \begin{pmatrix} -y + 1 \\ 0 \end{pmatrix}$$

Greens thm:

$$\int_\Gamma F \cdot dl = \oint_\Gamma -y\,dx + \underbrace{\oint_\Gamma 1\,dx}_{0}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$G = \begin{pmatrix} -y/2 \\ x/2 \end{pmatrix} = F + \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$= \begin{pmatrix} -y \\ 0 \end{pmatrix} + \begin{pmatrix} y/2 \\ x/2 \end{pmatrix}$$

$$\partial_1 (\ )_2 - \partial_2 (\ )_1$$

$$= \partial_1 \left(\frac{x}{2}\right) - \partial_2 \left(\frac{y}{2}\right) = 0$$

$$\varphi = \frac{xy}{2}$$

$$\begin{pmatrix} y/2 \\ x/2 \end{pmatrix} = \begin{pmatrix} \partial_x \varphi \\ \partial_y \varphi \end{pmatrix} = \begin{pmatrix} \partial_x \left(\frac{xy}{2}\right) \\ \partial_y \left(\frac{xy}{2}\right) \end{pmatrix}$$

Problem:

$$F = \begin{pmatrix} -y \\ 0 \end{pmatrix}. \qquad \partial_1 F_2 - \partial_2 F_1 = 1.$$

Suppose G is some fn $\ni$ $\partial_1 G_2 - \partial_2 G_1 = 1$

Q: Must $F - G$ be a potential force??

↳ A: Yes: $\partial_1 (F_2 - G_2) - \partial_2 (F_1 - G_1) = 1 - 1 = 0 \Rightarrow F - G$ is a pot!!

If $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, then $\operatorname{curl}(F) = \partial_1 F_2 - \partial_2 F_1$

Reason: Let $\tilde{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$\tilde{F}(x_1, x_2, x_3) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \\ 0 \end{pmatrix}$$

$$\nabla \times \tilde{F} = \begin{pmatrix} \partial_2 \tilde{F}_3 - \partial_3 \tilde{F}_2 \\ \sim \\ \partial_1 \tilde{F}_2 - \partial_2 \tilde{F}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

Cons / Pot / irr in 2D:

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is C^1

1 F is cons (i.e. $\oint_\Gamma F = 0$)

$\Leftrightarrow$ 2 F is pot (ie $\exists \varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t. $F = \nabla\varphi$)

$\Leftrightarrow$ 3 F is irr ($\partial_1 F_2 - \partial_2 F_1 = 0$).

3 $\Rightarrow$ 1 : Green's thm.:

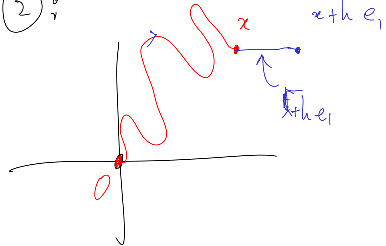
$$\underbrace{\oint_\Gamma F \cdot d\ell}_{\displaystyle \overset{|||}{0}} = \underbrace{\int_{\text{region encl by } \Gamma} (\partial_1 F_2 - \partial_2 F_1)\, dA}_{\displaystyle \overset{|||}{0}}$$

② $\Rightarrow$ ① : ~~If~~ Knows $\exists\, \varphi$ s.t. $F = \nabla \varphi$

$$\Rightarrow \oint_\Gamma F \cdot dl = \oint_\Gamma \nabla\varphi \cdot dl = \varphi(\text{end pt}) - \varphi(\text{start pt})$$

$\underbrace{}_{0}$ equal on a loop

① $\Rightarrow$ ② :



Define $\varphi(x) = \int_{\Gamma_x} F \cdot dl$

Γ_x = any path joining 0 & x

NTS $\nabla\varphi = F$.

1. Show $\partial_1 \varphi = F_1$

$$\partial_1 \varphi(x) = \lim_{h\to 0} \frac{\varphi(x+he_1) - \varphi(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_{\Gamma_{x+he_1}} F\cdot dl - \int_{\Gamma_x} F\cdot dl\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \underbrace{\int F\cdot dl}_{\text{blue piece}} = \lim_{h\to 0} \frac{1}{h}\int_0^h F(x+te_1)\cdot e_1\, dt$$

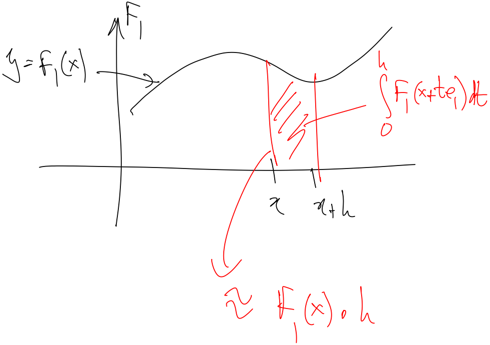
$$= \lim_{h\to 0} \frac{1}{h}\int_0^h F_1(x+te_1)\, dt$$

Param blue piece: $\gamma(t) = x + the_1 \quad (t\in(0,1))$

$\Leftrightarrow \quad \delta(t) = x + te_1 \quad (t \in (0,h))$

$$= F_1(x) \quad \text{(FTC)}$$

QED.


F_1
$y = F_1(x)$
x
$x+h$
$\int_0^h F_1(x+te_1)\,dt$
$\approx F_1(x) \cdot h$