Q4b (Optional HW)

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, C^1

$$\left| f(x)\, u(x) \right| < \frac{C}{|x|^{2+\varepsilon}} \qquad \varepsilon > 0 , \quad |x| > 1$$

$$\Rightarrow \int_{\mathbb{R}^3} \nabla f \cdot u \, dV = - \int_{\mathbb{R}^3} f (\nabla \cdot u) \, dV$$

Q4a) $U \subseteq \mathbb{R}^3$ bdd

$$\boxed{\int_U \nabla f \cdot u \, dV = \int_{\partial U} (f u) \cdot \hat{n} \, dS - \int_U f (\nabla \cdot u) \, dV}$$

Int by parts (1D) $\displaystyle\int_a^b f' g = [fg]_a^b - \int_a^b f g'$

for R

Pf of 4a: $$\underbrace{\int_U \nabla\cdot(f u)}_{\int_{\partial U} f u \cdot \hat{n}\, dS} = \int_U \nabla f \cdot u + \int_U f(\nabla\cdot u) \quad \text{QED}$$

$$\nabla\cdot(f u) = \sum \partial_i (f u_i)$$
$$= \sum \partial_i f\, u_i + f\, \partial_i u_i$$
$$= \nabla f\cdot u + f(\nabla\cdot u)$$

$$\boxed{\nabla\cdot(f u) = \nabla f\cdot u + f\, \nabla\cdot u}$$

4b:

$$\int_{\mathbb{R}^3} \nabla f \cdot u \, dV = \lim_{R\to\infty} \int_{B(0,R)} \nabla f \cdot u \, dV$$

$$= \lim_{R\to\infty} \Big(\underbrace{\int_{\partial B(0,R)} f u \cdot \hat{n} \, dS}_{} - \underbrace{\int_{B(0,R)} f \, \nabla\cdot u \, dV}_{\xrightarrow{R\to\infty} -\int_{\mathbb{R}^3} f (\nabla\cdot u) \, dV} \Big)$$

$$\left| \int_{\partial B(0,R)} f u \cdot \hat{n} \, dS \right| \leq \int_{\partial B(0,R)} \frac{C}{R^{2+\varepsilon}} \, dS$$

$|f u \cdot \hat{n}| \leq |f u|$

$$\leq \frac{C \, 4\pi R^2}{R^{2+\varepsilon}} \xrightarrow[R\to\infty]{} 0$$

Q3 b) Show $\underbrace{\int_U \nabla\cdot v}$ $\neq \int_{\partial U} v\cdot\hat{n}$ $\quad U = \{z > x^2 + y^2\}$

$v = (0, 0, e^{-z})$

$$\int_U \nabla\cdot v \, dV = \int_U -e^{-z}\, dV = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x^2+y^2}^{\infty} e^{-z}\, dz \, dx\, dy$$

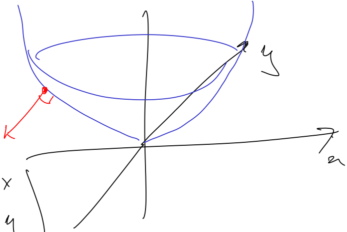
$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy \underset{\text{Polar}}{=} -\frac{1}{2}\int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, 2r \, d\theta\, dr$$

$(dx\,dy \to r\,d\theta\,dr)$

$$= -\pi$$

② $\int_{\partial U} v\cdot\hat{n}\, dS$:

Param ∂U : $\varphi(x,y) = \begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}$

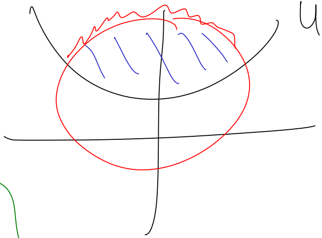


$$\partial_1\varphi \times \partial_2\varphi = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \hat{n}\circ\varphi = - \frac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|}$$

$$\Rightarrow \int_{\partial U} v\cdot\hat{n} = -\int_{\mathbb{R}^2} \begin{pmatrix} 0 \\ 0 \\ e^{-(x^2+y^2)} \end{pmatrix} \cdot \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} dx\,dy$$

$$= -\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy$$



$$w(s) = \frac{1}{2\pi} \int \frac{\rho(s,t)^{\perp} \cdot \partial_t \rho(s,t)}{|\rho(s,t)|^2}\, dt \in \mathbb{Z}$$

$\boxed{w \text{ is a cts fn of } s}$

$\to$ (1) Conservative : $\oint_\Gamma F \cdot dl = 0$

(2) Potential : $\exists \varphi \ni F = \nabla \varphi$ $\quad \left(\exists V \ni F = -\nabla V \right)$

(3) Irrotational : $\nabla \times F = 0$

On $\mathbb{R}^3$: (1) $\iff$ (2) $\iff$ (3)

easiest to check!

$U \subset \mathbb{R}^3$

Eg : $F(x, y, z) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$

$\mathbb{R}^3 - \{z \text{ axis}\}$

$$\boxed{U = \{(x,y,z) \mid x^2+y^2 > 0\}}$$

Claim F is irrotational: $(\nabla \times F = 0)$

$$\nabla \times F = \begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix} \times \begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix} = \begin{pmatrix}\partial_2 F_3 - \partial_3 F_2\\ \partial_3 F_1 - \partial_1 F_3\\ \partial_1 F_2 - \partial_2 F_1\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Claim F is NOT potential & F is NOT cons.

Pf: $\gamma(t) = \begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}$ $\qquad \oint_\Gamma F \cdot dl = +2\pi \neq 0$

$\Rightarrow F$ is NOT cons.

<u>Claim:</u> F is not potential

(Pf: If $\exists \varphi$? $F = \nabla\varphi \Rightarrow \oint_\Gamma \nabla\varphi \cdot dl = \varphi(b) - \varphi(a) = \underset{\text{cont}}{0}$)

What went wrong?

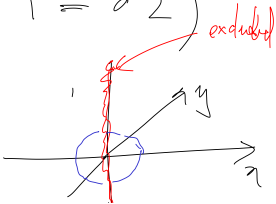
domain of $(F) = U \quad (\neq \mathbb{R}^3)$

For cons $\Leftrightarrow$ pot $\Leftrightarrow$ irr need the domain U to be simply connected

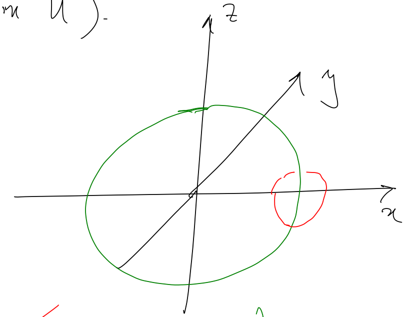
(i.e. If $\Gamma \subseteq U$ is any simple closed curve, then $\exists$ a surface $\underline{\Sigma \subseteq U}$ $\ni$ $\Gamma = \partial \Sigma$)

$U = \mathbb{R}^3 - \{z \text{ axis}\}$ is NOT simply conn.

(a circle of radius 1 in the xy plane & center O



does not bound a surface contained in U).

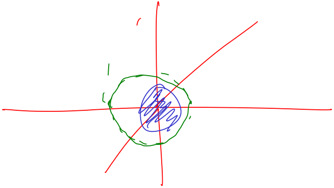


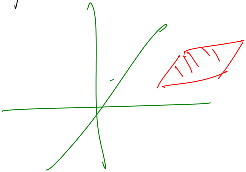
$V = \mathbb{R}^3 - \{\text{green ring}\}$

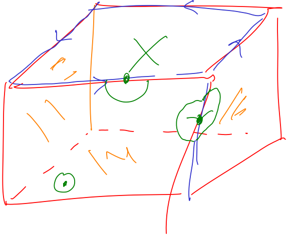
V is not simply connected

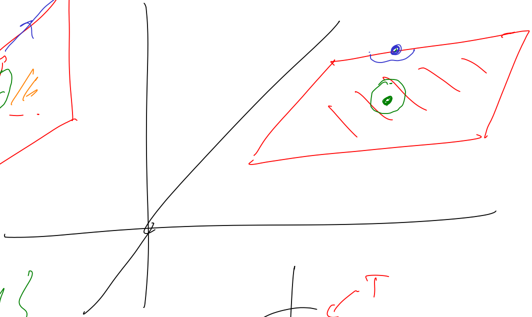
$W = \mathbb{R}^3 - B(0,1)$

is simply conn.



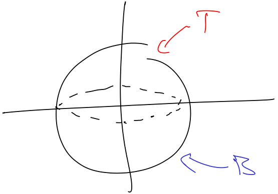






$$\Sigma = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$$

$$\partial \Sigma = \emptyset$$



$U \subseteq \mathbb{R}^3$ open

$\partial U \to \subseteq \mathbb{R}^3$ surface

$\partial(\partial U) = \emptyset$

$U \subset \mathbb{R}^3$

$F : U \to \mathbb{R}^3$.

$$\int_U \nabla \cdot F \, dV = \int_{\underbrace{\partial U}} \underbrace{F} \cdot \hat{n} \, dS$$

$$\int_U \underbrace{\nabla \cdot (\nabla \times G)} \, dV \xrightarrow[\text{div thm}]{} \int_{\underbrace{\partial U}_{\Sigma}} \nabla \times G \cdot \hat{n} \, dS \underset{\text{Stokes thm}}{=\!=} \oint_{\underbrace{\partial \Sigma}_{\phi}} G \cdot dl = 0$$

$\nabla \cdot \nabla \times G = 0 \quad \forall G$ fnc G

$$\int_\Sigma \nabla \times G \cdot \hat{n} \, dS = \int_{\partial \Sigma} G \cdot dl$$

$$\nabla \cdot \nabla \times G = \nabla \cdot \left[\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} \right] = \nabla \cdot \begin{pmatrix} \partial_2 G_3 - \partial_3 G_2 \\ \partial_3 G_1 - \partial_1 G_3 \\ \partial_1 G_2 - \partial_2 G_1 \end{pmatrix}$$

$$= \cancel{\partial_1 \partial_2 G_3} - \cancel{\partial_1 \partial_3 G_2} + \cancel{\partial_2 \partial_3 G_1} - \cancel{\partial_2 \partial_1 G_3} + \cancel{\partial_3 \partial_1 G_2} - \cancel{\partial_3 \partial_2 G_1} = 0$$