① Final → rate on time.
② Fill out FCE's → Incentive: $\boxed{75\%}$ response rate → Grades Early
If not → Grades at the very very end of grading period. (May 19).

Next week: OH as usual (Mon & today)
OH on Wed (poll)
Recitation on Tue.
Final on Thu

Fundamental thm of algebra: If $f: \mathbb{C} \to \mathbb{C}$ is a polynomial of degree ≥ 1, then f has a root in $\mathbb{C}$.

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R},\ i = \sqrt{-1} \}$$

Pf using cont & winding #: (1) $\mathbb{C} \equiv \mathbb{R}^2$ (Complex plane)

$z \in \mathbb{C}$, $z = x + iy$

identify $\underset{\mathbb{C}}{\underset{\cap}{z}} \equiv (x, y) \in \mathbb{R}^2$.

(2) Alg of $\mathbb{C}$.

$$\boxed{(3)\ e^{i\theta} = \cos\theta + i \sin\theta}$$

Step 1: Winding #: $\Gamma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a closed curve which doesn't pass through 0, then

$$W(\Gamma) = \frac{1}{2\pi} \oint_\Gamma \frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{1}{2\pi} \int_0^1 \frac{\gamma(t)^\perp \cdot \gamma'(t)}{|\gamma(t)|^2}\,dt$$

where γ is a param of Γ & $\gamma(t)^\perp = \begin{pmatrix} -\gamma_2(t) \\ \gamma_1(t) \end{pmatrix}$

$\left(u \in \mathbb{R}^2,\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ u^\perp = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} = u \text{ rotated by } 90^\circ \right).$

$W(\Gamma) \in \mathbb{Z}$ $\forall$ closed curves that don't pass through 0.

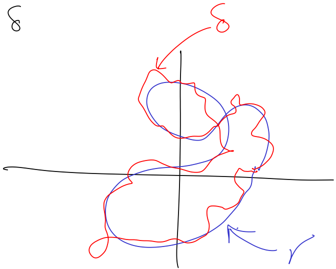
Step 2: If γ & δ are 2 closed parametric curves (that don't pass through O)

such that γ can be cts-ly deformed to δ

(as a loop, without passing through O)

then $W(\gamma) = W(\delta)$

(Rigorous way $\to$ loop homotopies)



Eg 1: $\gamma : [0,1] \to \mathbb{R}^2 - \{0\}$, C^1

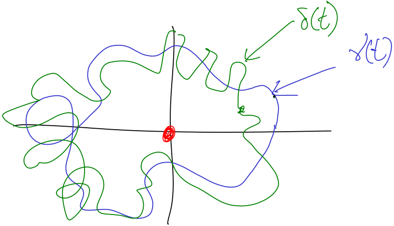
$\delta : [0,1] \to \mathbb{R}^2 - \{0\}$, C^1

$\gamma(0) = \gamma(1)$ & $\delta(0) = \delta(1)$

Suppose $|\delta(t) - \gamma(t)| < |\gamma(t)|$

then $W(\gamma) = W(\delta)$

Pf: Invent a new variable s (def variable)



Define $\rho(s,t) = \gamma(t) + s(\delta(t) - \gamma(t))$

Note: $\rho(0,t) = \gamma(t)$

$\rho(1,t) = \delta(t)$

$\forall s \in [0,1]$, consider $\rho(s,t)$ as a param curve in t (fixed s)

Q: ① Is this curve a loop $\rightarrow$ Yes $(\rho(s,0) = \rho(s,1))$

② Does it pass though 0? No!

$|\rho(s,t)| \geq |\gamma(t)| - s|\delta(t) - \gamma(t)| > 0$ by assumption.

$\left(\because \gamma(t) - \rho(s,t) = s\left(\gamma(t) - \delta(t)\right) \right)$

Let $w(s)$ = winding # of the curve $\rho(s,t)$ (as a fn of t).

$$= \frac{1}{2\pi} \int_{t=0}^{1} \frac{\rho(s,t)^{\perp} \cdot \partial_t \rho(s,t)}{|\rho(s,t)|^2} \, dt$$

$\Rightarrow w(s)$ is a cts fn of s. & $w(s) \in \mathbb{Z}$

$\Rightarrow w(0) = w(1) \Rightarrow W(\gamma) = W(\delta)$ QED!

Fund thm of alg: let $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$

$a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$.

Compute $\lim_{|z| \to \infty} \frac{|f(z) - z^n|}{|z|^n} = \lim_{|z|\to\infty} \frac{|\cancel{z^n} + \sum_{i<n} a_i z^i|}{|z|^n} = \cancel{1}\, 0$

$$\cancel{\Rightarrow} \left(\frac{|z^n + \sum_{i<n} a_i z^i|}{|z|^n} \leq \underbrace{\frac{|z|^n}{|z|^n}}_{= 1} + \sum_{i<n} \underbrace{\frac{|a_i|}{|z|^{n-i}}}_{\xrightarrow[|z|\to\infty]{} 0} \right)$$

$$\Rightarrow \lim_{|z|\to\infty} \frac{|f(z)|}{|z|^n} = 1. \quad \Rightarrow \exists R_0 \ni |z| \geq R_0 \Rightarrow |f(z) - z^n| < |z|^n$$

$$\boxed{\Rightarrow |f(z) - z^n| < |z|^n.}$$

Let $\delta(t) = R_0^n e^{2\pi i n t} = \begin{pmatrix} R_0^n \cos 2\pi n t \\ R_0^n \sin 2\pi n t \end{pmatrix} = (\underbrace{R_0 e^{2\pi i t}}_{z})^n$

& $\gamma(t) = f(\underbrace{R_0 e^{2\pi i t}}_{z}).$

From $\square$ know $|\delta(t) - \gamma(t)| < |\delta(t)|$

$\Rightarrow$ Lemma $w(\delta) = w(\gamma) = \underline{\underline{n}}$ (can compute).

Claim: If f has no roots (i.e. $f(z) \neq 0 \ \forall z \in \mathbb{C}$) then $w(\gamma) = 0$. [This gives a contradiction & finishes the proof].

Pf of Claim:

Let $\rho(s,t) = f\left(s R_0 e^{2\pi i t}\right)$

When $s=1$, $\rho(s,t)$ traces out $\gamma(t)$ (as t varies).

when $s=0$, $\rho(s,t) = f(0)$ const.

Note: $\rho(s,0) = \rho(s,1)$ (from formula)

& $\rho(s,t) \neq 0 \ \forall s, t$ (since f has no roots by assump)

$\Rightarrow$ ~~wind~~ $W(\gamma)$ = winding # of $\rho(0,t)$ (as a fn of t)

$= 0 \neq n$

QED :)