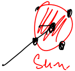
Quick Eg:



Sun

O (mass M)

x (mass m) $\quad (x \in \mathbb{R}^3)$

Big ball of gas

Q: Gravitational force

→ radially inward

→ prop to $M \cdot m$

→ ~~inv~~ prop to $\dfrac{1}{|x|^2}$

$$F(x) = -G \frac{Mm}{|x|^2} \underbrace{\left(\frac{x}{|x|} \right)}_{\text{unit vector pointing radially } \cancel{\text{inward}} \text{ outward}}$$

G: Grav const.

$$F(x) = -\frac{GMm}{|x|^2}\frac{x}{|x|} = -GMm\frac{x}{|x|^3}.$$

(Why) Yes No

Q: (i) At time 0 assume velocity 0

① As the ball is pulled towards the sun does it spin?

② " " " " " " " does it shrink/expand. → Yes No (Why)

Maybe: ~~Rotation~~ Spin compute $\nabla \times F$

Volume exp compute $\nabla \cdot F$

∇	→ grad
$\nabla \times$	→ curl
$\nabla \cdot$	→ div

$\nabla \cdot F$: Note $F(x) = -\nabla V$, where

$$V = \frac{GMm}{|x|}$$

(like the fn N from exam HW).

Same calc as HW: $\nabla \cdot F = \underbrace{\nabla \cdot \nabla}_{\Delta} V = 0$

Δ (Laplacian) or ∇^2.

$$\left[\nabla \cdot \nabla f = \nabla \cdot \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f = \Delta f. \right]$$

$\Rightarrow \nabla \cdot F = 0 \Rightarrow$ No volume expn/contraction.

Spin: $\nabla \times F$: Note V is C^2

Claim: For any C^2 fn $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, we have $\nabla \times (\nabla \varphi) = 0$

$$\left(\text{Pf: } \nabla \times \nabla \varphi = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} \partial_1 \varphi \\ \partial_2 \varphi \\ \partial_3 \varphi \end{pmatrix} = \begin{pmatrix} \partial_2 \partial_3 \varphi - \partial_3 \partial_2 \varphi \\ \text{un} \\ \text{un} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\boxed{\Rightarrow \nabla \times \nabla \varphi = 0 \quad \forall\, C^2 \text{ fns } \varphi}$$

$\Rightarrow$ For Grav force $F = -\nabla V \rightarrow \nabla \times F = 0 \Rightarrow$ no spin!

Conservative / potential forces: (1) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a cons force if:

$\forall a, b, \ \forall \Gamma_1, \Gamma_2$ joining a & b, $\int_{\Gamma_1} F \cdot dl = \int_{\Gamma_2} F \cdot dl$

(line int is independent of the path taken)

(2) We say F is a potential force if $\exists V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t. $F = -\nabla V$

$(\Leftrightarrow \exists \varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t. $F = \nabla \varphi)$.

(3) We say F is <u>irrotational</u> if $\nabla \times F = 0$ easy to check.

Thm: On $\mathbb{R}^3$, (1) F is cons $\Leftrightarrow$ (2) F is potential $\Leftrightarrow$ (3) F is irrotational.

(Long time ago: Showed F potential $\Rightarrow$ F is cons. $\left(\because F = \nabla\varphi \Rightarrow \int_\Gamma F \cdot d\ell = \varphi(b) - \varphi(a)\right.$ ind of path$\left.\right)$,

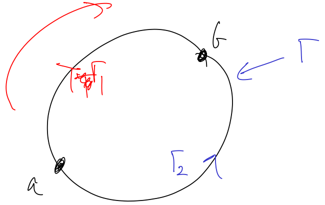
Just now: Showed F potential $\Rightarrow$ F is irrotational. $(\because \nabla \times \nabla\varphi = 0)$.

Pf: Know (2) $\Rightarrow$ (1) & (2) $\Rightarrow$ (3)

Lets show (3) $\Rightarrow$ (1) (irrotational $\Rightarrow$ cons):

Remark: F is cons $\Leftrightarrow$ $\forall$ closed loops Γ, $\oint_\Gamma F \cdot d\ell = 0$

Pf of Rmk: Say Γ is a closed loop.



$$\oint_\Gamma F\cdot dl = \int_{\Gamma_1} F\cdot dl - \int_{\Gamma_2} F\cdot dl$$

$= 0$ since F is cons!

(converse: reverse the steps).

Pf of ③ $\Rightarrow$ ①: (Irr $\Rightarrow$ cons). Will show $\oint_\Gamma F\cdot dl = 0$ $\forall$ closed loops Γ

Pf: let Σ be any surface whose bdry is Γ

Stokes thm $\Rightarrow$ $\oint_\Gamma F\cdot dl = \int_\Sigma \nabla\times F\cdot\hat{n}\, dS = 0$

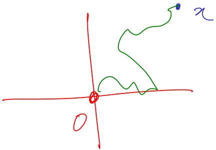
(① ⇒ ③ can also be done quickly via Stokes Thm)

Pf of ① ⇒ ②. (Cons ⇒ potential)

Assume $\oint F \cdot d\ell = 0$ NTS $\exists \varphi \ni F = \nabla \varphi$.

Idea: Let $\varphi(x) = \int_{\Gamma_x} F \cdot d\ell$, where Γ_x = any path joining 0 & x

(well def because F is cons)



Claim: $\nabla \varphi = F$

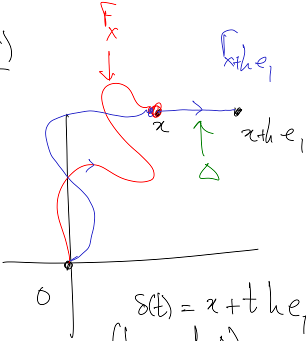
Finish show $\partial_1 \varphi(x) = F(x)$

Let $h > 0$ $\quad \partial_1 \varphi(x) = \lim_{h \to 0} \dfrac{\varphi(x + h e_1) - \varphi(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_{\Gamma_{x+he_1}} F \cdot dl - \int_{\Gamma_x} F \cdot dl \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{\triangle} F \cdot dl = \frac{1}{h} \int_{\triangle} F \circ \delta \cdot \delta' \, dt$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^1 F_1(x + t h e_1) \, h \, dt$$



$\delta(t) = x + t h e_1$
(param of $\triangle$)

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h F_1(x + s e_1)\, ds = F_1(x)$$ QED!!

Remark 1: Showed Cons $\Leftrightarrow$ pot $\Leftrightarrow$ irr on $\mathbb{R}^3$
Also true on domains U, provided U is "simply connected"
(i.e. every closed loop in U bounds a surface that is completely contained in U).

Remark 2: Note $\nabla \times (\nabla \varphi) = 0$. & we showed $\nabla \times F = 0 \Leftrightarrow F = \nabla \varphi$
Can also check: $\nabla \cdot (\nabla \times F) = 0$. Q: If $\nabla \cdot u = 0$, does $\exists F \ni u = \nabla \times F$
(Yes, try it)