21-268: Multidimensional Calculus **Spring 2020**

Recitation April 28

Stokes Theorem

Recall the definition of the curl of a vector field $\mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{curl}(F) = \nabla \times F := \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

The Stokes Theorem connects the surface integral of $\nabla \times F$ with line integral of F at the boundry.

Theorem 1 *Let $U \subset \mathbb{R}^3$ be a domain, $(\Sigma, \hat{n}) \subset U$ be a bounded, oriented, piecewise C^1 surface whose boundary is a piecewise C^1 curve Γ. If $F : U \to \mathbb{R}^3$ is a C^1 vector field, then*

$$\int_\Sigma \nabla \times F \cdot \hat{n} dS = \oint_\Gamma F \cdot dl$$

The line integral is calculated counter-clockwise (w.r.t. $\hat{n}$) for outer boundary and clockwise for inner boundary (holes)

Example 1

Let $\Sigma = \{(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 \leq 1\}$, $F : \mathbb{R}^3 \to \mathbb{R}^3$ be

$$F(x, y, z) = (z, x, y)$$

Compute

$$\int_\Sigma \nabla \times F \cdot \hat{n}\, dS$$

where $\hat{n}$ is the upward unit normal

Solution Let $\Gamma = \partial\Sigma = \{(x, y, 0) : x^2 + y^2 = 1\}$ (the unit circle with $z = 0$).

Since Γ, F, Σ are obviously C^1, we can use the Stokes theorem and use the parametrization

$\gamma : [0, 2\pi] \to \Gamma$, $\gamma(t) = (\cos t, \sin t, 0)$

$$\begin{aligned}
\int_\Sigma \nabla \times F \cdot \hat{n}\, dS &= \int_\Gamma F \cdot dl \\
&= \int_0^{2\pi} \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt \\
&= \int_0^{2\pi} \cos^2 t dt \\
&= \int_0^{2\pi} \frac{1}{2}(1 + \cos(2t))dt \\
&= \left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]\Bigg|_{t=0}^{2\pi} \\
&= \pi
\end{aligned}$$

Another way to compute it is that notice that Γ is also the boundary of the disk $\Sigma' := \{(x, y, 0) : x^2 + y^2 \leq 1\}$, then applying Stokes twice we have

$$\begin{aligned}
\int_\Sigma \nabla \times F \cdot \hat{n}\, dS &= \int_\Gamma F \cdot dl \\
&= \int_{\Sigma'} \nabla \times F \cdot \hat{n}\, dS \\
&= \int_{\Sigma'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS \\
&= \int_{\Sigma'} 1\, dS \\
&= \text{Area}(\Sigma') = \pi
\end{aligned}$$

Remark: Be careful with orientation.

Example 2 In this example we will prove the following lemma.

Let $\Sigma \subset \mathbb{R}^3$ be a C^1 bounded oriented surface and and $\partial\Sigma = \Gamma$ be a closed C^1 curve in $\mathbb{R}^3$, $f : \mathbb{R}^3 \to \mathbb{R}$ a C^2 function, then $\int_\Gamma \nabla f \cdot dl = 0$

Proof: Let's prove this in two ways.

First we use the Stokes theorem.

Since $f \in C^2$,

$$\nabla \times \nabla f = \begin{pmatrix} \partial_2\partial_3 f - \partial_3\partial_2 f \\ \partial_3\partial_1 f - \partial_1\partial_3 f \\ \partial_1\partial_2 f - \partial_2\partial_1 f \end{pmatrix} = 0$$

So by Stokes theorem,

$$\int_\Gamma \nabla f \cdot dl = \int_\Sigma 0 \cdot \hat{n}\, dS = 0$$

The second way is the fundamental theorem of line integral, can any point a on Γ, then the fundamental theorem of line integral tells us

$$\int_\Gamma \nabla f \cdot dl = f(a) - f(a) = 0$$

(This actually only need $f \in C^1$ and Γ)

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Divergence Theorem

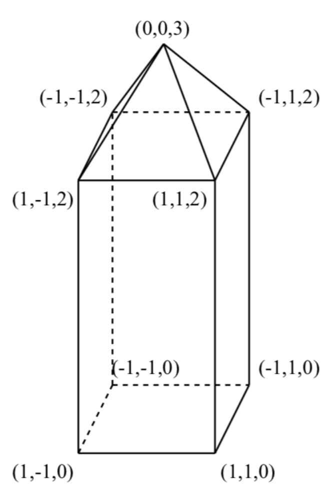
Let $U \subset \mathbb{R}^3$ be open and $F : U \to \mathbb{R}^3$ be a C^1 vector field. The divergence of F is

$$\operatorname{div}(F) = \nabla \cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

Theorem 2 *Let $U \subset \mathbb{R}^3$ be a bounded region such that ∂U is a piecewise C^1 surface. Let $F : \overline{U} \to \mathbb{R}^3$ be a C^1 vector field and $\hat{n}$ the outward pointing normal vector on ∂U. Then,*

$$\int_U \nabla \cdot F \,\mathrm{d}V = \int_{\partial U} F \cdot \hat{n}\,\mathrm{d}S$$

Example 1: Compute $\int_\Sigma F \cdot \hat{n}\,\mathrm{d}S$, where $F(x, y, z) = (xz + 3, -268y, -\frac{1}{2}z^2)$, $\hat{n}$ is an inward pointing normal vector, and Σ is the surface of the following object:



Solution: We note that Σ is a closed surface. Let U be the region enclosed by Σ. By divergence theorem, we have

$$\begin{aligned}
\int_\Sigma F \cdot \hat{n} \, \mathrm{d}S &= -\int_\Sigma F \cdot (-\hat{n}) \, \mathrm{d}S \\
&= -\int_U \nabla \cdot \begin{pmatrix} xz+3 \\ -268xy \\ -\frac{1}{2}z^2 \end{pmatrix} \mathrm{d}V \\
&= \int_U 268x \, \mathrm{d}V \\
&= \int_0^2 \int_{-1}^1 \int_{-1}^1 268x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z + \int_2^3 \int_{z-3}^{3-z} \int_{z-3}^{3-z} 268x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \quad \text{(Fubini)} \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Example 2: Let $U \subset \mathbb{R}^3$ be a bounded region such that ∂U is a piecewise C^1 surface. Let $u : \overline{U} \to \mathbb{R}$ be a C^2 function. Show that if u is harmonic ($\Delta u = 0$), then $\int_{\partial U} \nabla u \cdot \hat{n} \, \mathrm{d}S = 0$.

Solution: Note that

$$\nabla \cdot \nabla u = \sum_{i=1}^3 \partial_i(\partial_i u) = \sum_{i=1}^3 \partial_i^2 u = \Delta u$$

Thus, by divergence theorem,

$$\int_{\partial U} \nabla u \cdot \hat{n} \, \mathrm{d}S = \int_U \nabla \cdot \nabla u \, \mathrm{d}V = \int_U \Delta u \, \mathrm{d}V = 0$$

as desired.

Example 3: (Green's first identity)

Let $U \subset \mathbb{R}^3$ be a bounded region such that ∂U is a piecewise C^1 surface. Let $f, g : \overline{U} \to \mathbb{R}$ be C^2 functions. Then,

$$\int_U \nabla f \cdot \nabla g \, \mathrm{d}V = -\int_U f \Delta g \, \mathrm{d}V + \int_{\partial U} f \nabla g \cdot \hat{n} \, \mathrm{d}S$$

Solution: Note that

$$\nabla \cdot (f \nabla g) = \sum_{i=1}^3 \partial_i (f \partial_i g) = \sum_{i=1}^3 \partial_i f \partial_i g + f \partial_i^2 g = \nabla f \cdot \nabla g + f \Delta g$$

Thus, by divergence theorem,

$$\int_{\partial U} f\nabla g \cdot \hat{n}\,\mathrm{d}S = \int_U \nabla \cdot (f\nabla g)\,\mathrm{d}V = \int_U \nabla f \cdot \nabla g\,\mathrm{d}V + \int_U f\Delta g\,\mathrm{d}V$$

as desired.