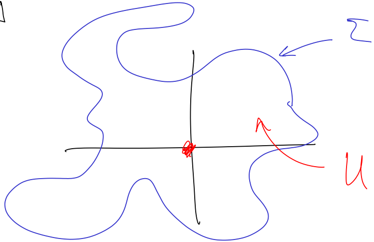
Q 5 : $N(x) = \dfrac{-1}{4\pi |x|}$ $x \in \mathbb{R}^3 - \{0\}$

Goal : $\displaystyle\int_\Sigma \nabla N \cdot \underbrace{\hat{n}}_{\text{outward pointing unit normal}} \, dS$

$\Sigma \rightarrow$ any closed surface that contains O

U = region encl by Σ



Try div thm :

If it applied then

$$\int_\Sigma \nabla N \cdot \hat{n} \, dS = \int_U (\underbrace{\nabla \cdot}_{\text{div}} \underbrace{\nabla N}_{\text{grad } N}) \, dV$$

Compute $\nabla\cdot(\nabla N) = \frac{-1}{4\pi}\nabla\cdot\left(\nabla\left(\frac{1}{|x|}\right)\right)$ $\longrightarrow$ $\frac{-1}{|x|^3}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$= \frac{-1}{4\pi}\nabla\cdot\left(\frac{-1}{|x|^2}\,\frac{x}{|x|}\right)$$

$$= \frac{+1}{4\pi}\nabla\cdot\begin{pmatrix} x_1/|x|^3 \\ x_2/|x|^3 \\ x_3/|x|^3 \end{pmatrix} \rightsquigarrow \boxed{\text{simplifies to } 0 \text{ (You check)}}$$

$\nabla \longrightarrow$ grad

$\nabla\times \longrightarrow$ curl

$\nabla\cdot \longrightarrow$ div.

$$\Rightarrow \int_{\Sigma} \nabla N\cdot\hat{n}\, dS = \int_{U} \nabla\cdot\nabla N\, dV = \int_{U} 0\, dV = 0$$

Won't work because $0 \in U$ & N is not defined at 0

Note ① If $0 \notin U$, then the above works &

$$\int_{\partial\Sigma} \nabla N \cdot \hat{n} = 0$$ (because div thm applies)

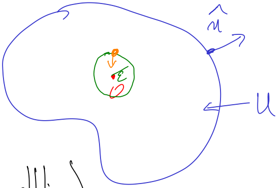
② If $0 \in U$: "Ball trick"

ⓐ Choose $\varepsilon > 0$ s.t. $B(0,\varepsilon) \subseteq U$

ⓑ $V_\varepsilon = U - B(0,\varepsilon)$

ⓒ $$\int_{\partial V_\varepsilon} \nabla N \cdot \hat{n} \, dS = \int_{V_\varepsilon} (\nabla \cdot \nabla N) \, dV = 0$$

(div thm applies)



(d) $\partial V_\varepsilon = \partial U \cup \partial B(0,\varepsilon)$

$$\Rightarrow \underbrace{\int_{\partial V_\varepsilon} \nabla w \cdot \hat{n}\, dS}_{=0} = \boxed{\int_{\partial U} \nabla w \cdot \underbrace{\hat{n}}_{\substack{\text{points outward}\\ \text{to } V_\varepsilon\\ (\text{outward to } U)}} dS} + \int_{\partial B(0,\varepsilon)} \nabla w \cdot \underbrace{\hat{n}}_{\substack{\text{outward to } V_\varepsilon\\ \boxed{\text{inward to } \partial B(0,\varepsilon)}}} dS$$

$$\Rightarrow \int_{\partial U} \nabla w \cdot \hat{n}\, dS = -\int_{\partial B(0,\varepsilon)} \nabla w \cdot \underbrace{\hat{n}}_{\text{radially inward}} dS = +\int_{\partial B(0,\varepsilon)} \nabla w \cdot \underbrace{\hat{n}}_{\text{radially outward}} dS$$

$$= \int_{\partial B(0,\varepsilon)} +\frac{1}{4\pi} \frac{x}{|x|^3} \cdot \frac{x}{|x|} \, dS$$

$$= \frac{1}{4\pi} \int_{\partial B(0,\varepsilon)} \frac{1}{|x|^2} \, dS = \frac{1}{4\pi} \int_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon^2} \, dS$$

$$= \frac{\text{area}(\partial B(0,\varepsilon))}{4\pi\varepsilon^2} = 1$$

$$\Rightarrow \int_\Sigma \nabla N \cdot \hat{n} \, dS = \begin{cases} 0 & \Sigma \text{ does not enclose } 0 \\ 1 & \Sigma \text{ does enclose } 0 \end{cases}$$

Q4: (a) A - 3×3. $u, v \in \mathbb{R}^3$

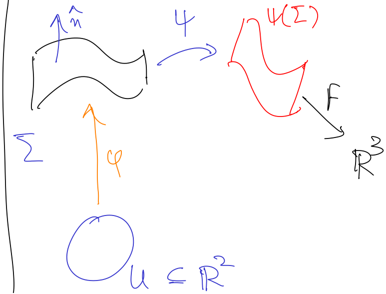
$$Au \times Av = \operatorname{adj}(A)^T (u \times v)$$

(Elegant way exists)

(b) $U \subset \mathbb{R}^3$ $(\Sigma, \hat{n})$ oriented surface $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ $\det(D\psi) > 0$

$F : \psi(\Sigma) \to \mathbb{R}^3$ is a ~~vector field~~ function

$$\int_{\psi(\Sigma)} F \cdot \hat{n} \, dS = \int_{\Sigma} \_\_\_\_\_\_\_\_\_\_\_\_$$



① Let φ be a param of ~~$\psi(\Sigma)$~~ Σ

Note $\psi\circ\varphi$ is a param of $\psi(\Sigma)$

$$\left(\frac{\partial_1\varphi\times\partial_2\varphi}{|\partial_1\varphi\times\partial_2\varphi|} = \pm\hat{n}\circ\varphi\right)$$

$$\boxed{\left(\frac{\partial_1(\psi\circ\varphi)\times\partial_2(\psi\circ\varphi)}{|\quad\times\quad|} = +\hat{n}\circ\psi\circ\varphi\right) \quad (\det D\psi > 0)}$$

② $$\int_{\psi(\Sigma)} F\cdot\hat{n}\,dS = \int_U F\circ(\psi\circ\varphi)\cdot\underbrace{\left(\partial_1(\psi\circ\varphi)\times\partial_2(\psi\circ\varphi)\right)}\,dA$$

$$\partial_1(\psi\circ\varphi) = \underbrace{D(\psi\circ\varphi)}\,e_1$$

$$= D\psi_\varphi\,D\varphi\,e_1 = \overbrace{D\psi_\varphi}^{A}\,\partial_1\varphi$$

$$\Rightarrow \partial_1(\psi\circ\varphi)\times\partial_2(\psi\circ\varphi) = \left((D\psi_\varphi)\partial_1\varphi\times(D\psi_\varphi)\partial_2\varphi\right)$$ & use 4a

$$\Rightarrow \int_{\psi(\Sigma)} F \cdot \hat{n} \, dS = \int_{U} (F \circ \psi) \circ \varphi \cdot \left[\operatorname{adj}(D\psi_{\varphi})^{T} \, \partial_1 \varphi \times \partial_2 \varphi \right] dA$$

$$= \int_{U} \operatorname{adj}(D\psi_{\varphi}) (F \circ \psi) \circ \varphi \quad \underbrace{\partial_1 \varphi \times \partial_2 \varphi \, dA}$$

$$= \int_{\Sigma} (\operatorname{adj} D\psi) \, F \circ \psi \cdot \hat{n} \, dS$$

(4a) NTS $Au \times Av = \operatorname{adj}(A)\, u \times v$

Say A is inv. Then enough to show $(Au \times Av) \cdot Aw = (\operatorname{adj}(A)^T u \times v) \cdot Aw$ $\forall w \in \mathbb{R}^3$

$$(Au \times Av) \cdot Aw = \det \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ Au & Av & Aw \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \det \left(A \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ u & v & w \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \right)$$

$$= \det(A)\, (u \times v) \cdot w$$

$$\left(\operatorname{adj}(A)^T u \times v \right) \cdot Aw = (u \times v) \cdot \left(\underbrace{\operatorname{adj}(A)\, A}_{} w \right) = (u \times v) w \, (\det A)$$

$\det(A)\, w$

Q.E.D.