Divergence Theorem: $U \subseteq \mathbb{R}^3$ $\boxed{\text{bdd}}$ domain

∂U is piecewise C^1 surface

$\hat{n}$ = outward pointing normal vector.

$F : \overline{U} \to \mathbb{R}^3$ is C^1. Then

$$\int_U (\nabla \cdot F)\, dV = \int_{\partial U} F \cdot \hat{n}\, dS$$

Reminder: $\nabla \longrightarrow$ grad $\left(\nabla f = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} \right)$

$\nabla \times \longrightarrow$ curl $\left(\nabla \times u = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \vdots \\ \end{pmatrix} \right.$

$\nabla \cdot \longrightarrow$ divergence $\left(\nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 \right)$

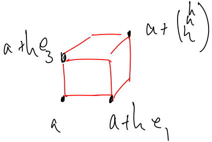
$$\downarrow \begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix}\cdot\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}$$

Intuition about divergence: Curl $\longrightarrow$ swirl

divergence $\longrightarrow$ volume expansion/compression.

$u(x)$ = velocity of a fluid at $x \in \mathbb{R}^3$ $\quad (u(x) \in \mathbb{R}^3)$.

Imagine a small box placed at a & transported by the fluid.

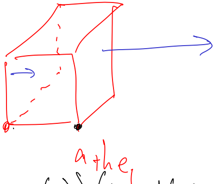


Compute the change in volume:

$V(t)$ = volume of box after time t (t small).

New x_1 side length $\approx \left(a_1 + he_1 + \underbrace{tu_1(a+he_1)} - (a_1 + tu_1(a))\right)$

$$= \left(h + t\left(u_1(a+he_1) - u_1(a)\right)\right)$$



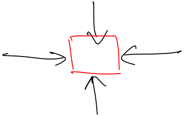
Change in volume $= \left(h + t(u_1(a+he_1) - u_1(a))\right)\left(h + t(u_2(a+he_2) - u_2(a))\right)\left(h + t(u_3(a+he_3) - u_3(a))\right) - h^3$

$$\approx \left(h + th\,\partial_1 u_1(a)\right)\left(h + th\,\partial_2 u_2(a)\right)\left(h + th\,\partial_3 u_3(a)\right) - h^3$$

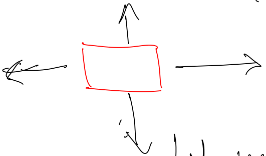
$$= t\,h\,(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) + t^2 h^2 (\cdots)$$

$$\Rightarrow \underbrace{\frac{\text{change in volume}}{t}}_{\text{rate of change of vol.}} \quad \text{as } t \to 0 \quad \approx \underline{h}\,\boxed{(\nabla \cdot u)}$$

volume expansion



(sink) high -ve div.



high +ve div.

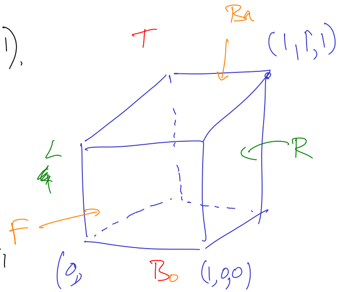
Pf of Div thm:

Part I: Check on a cube. (FTC & Fubini)

Part II: Coordinate change.

Part I: $C = (0,1)^3 = (0,1) \times (0,1) \times (0,1)$.

NTS $\underbrace{\int_C (\nabla \cdot F) \, dV}_{} = \int_{\partial C} F \cdot \hat{n} \, dS$



① $\int_C \nabla \cdot F \, dV = \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} (\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \, dx_3 \, dx_2 \, dx_1$

$$= \underbrace{\int_{x_3=0}^{1}\int_{x_2=0}^{1}\int_{x_1=0}^{1} \partial_1 F_1 \, dx_1 \, dx_2 \, dx_3}_{I_1} + \underbrace{\int_{x_1=0}^{1}\int_{x_3=0}^{1}\int_{x_2=0}^{1} \partial_2 F_2 \, dx_2 \, dx_3 \, dx_1}_{\leftarrow I_2} + \underbrace{\int_{0}^{1}\int_{0}^{1}\int_{x_3=0}^{1} \partial_3 F_3 \, dx_3 \, dx_1 \, dx_2}_{I_3}$$

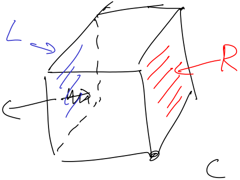
$$= I_1 + I_2 + I_3$$

Compute I_1:

$$I_1 = \int_{x_3=0}^{1}\int_{x_2=0}^{1} \left[F_1(1, x_2, x_3) - F_1(0, x_2, x_3) \right] dx_2 \, dx_3$$

Note On R (right face), $\hat{n} = e_1 \Rightarrow F \cdot \hat{n} = F_1$

On L (left face), $\hat{n} = -e_1$



$$\Rightarrow I_1 = \int_R F \cdot \hat{n} \, dS + \int_L F \cdot \hat{n} \, dS.$$

Do the same for I_2 & I_3 & get $\displaystyle\int_C \nabla \cdot F = \int_{\partial C} F \cdot \hat{n} \, dS$

Q.E.D.

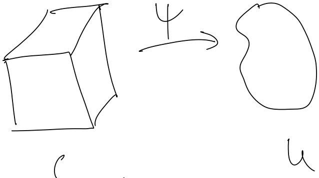
Part 2:

Say $U \subseteq \mathbb{R}^3$ is a nice domain.

& $\exists\, \varphi : \overline{C} \to \overline{U}$ s.t. φ is $\underline{\underline{C^2}}$, $\overbrace{\det(D\varphi) > 0}^{\text{orientation pres.}}$
bij

$\varphi : \partial C \to \partial U$

Let $F : \overline{U} \to \mathbb{R}^3$ be C^1



$$\int_{\partial U} F \cdot \hat{n}\, dS = \int_{\varphi(\partial C)} F \cdot \hat{n}\, dS$$

$$\text{(HW Q4b)} = \int_{\partial C} \operatorname{adj}(D\varphi) F \circ \varphi \cdot \hat{n}\, dS$$

(adj = adj matrix).

div thm on cube

$$\int_C \nabla \cdot \left(\operatorname{adj}(D\psi)\, F \circ \psi \right) dV$$

(compute via prod & chain rule)

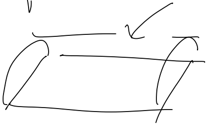
$$= \int_C \det(D\psi)\, (\nabla \cdot F) \circ \psi \; dV$$

coordinate change

$$\int_U (\nabla \cdot F)\, dV$$

QED.

Q3: $\vec{f}$



speed of rotation in the $x_1 - x_3$ plane $\approx 2h\,(\nabla \times u \cdot e_2)$

$h\,\Big(\overbrace{u_1(a-he_3) - u_1(a+he_3)} + u_3 \ldots\Big)$

$\underline{-\partial_3 u_1} \cdot 2h$

$f(a+h) - f(a) \approx h f'(a)$