



A diagram of a small surface element  $dS$  on a surface  $\Sigma$ . The element is shown as a parallelogram. A vertical green arrow labeled  $\hat{n}$  indicates the normal direction. Two other green arrows labeled  $\hat{u}$  and  $\hat{v}$  represent unit tangent vectors along the edges of the element. To the right, the formula for the surface integral of a scalar function  $\rho$  is given as  $\int_U \rho \cdot |\partial_1 \varphi \times \partial_2 \varphi| dA$ .

$$\textcircled{1} \quad \rho: \Sigma \rightarrow \mathbb{R} \quad \text{then} \quad \int_{\Sigma} \rho \, dS = \int_U \rho \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA.$$

$\textcircled{2} \quad F: \Sigma \rightarrow \mathbb{R}^3$  &  $\Sigma$  is oriented ( $\hat{n}$  is given)

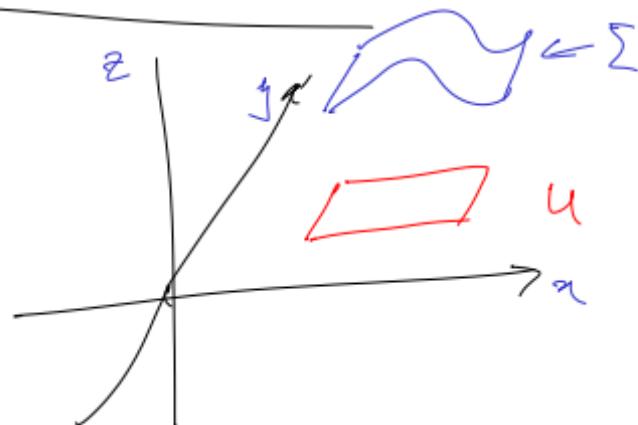
then  $\frac{\partial_1 \varphi \times \partial_2 \varphi}{|\partial_1 \varphi \times \partial_2 \varphi|}$  is a unit normal vec  $\Rightarrow \hat{n} = \pm \frac{\partial_1 \varphi \times \partial_2 \varphi}{|\partial_1 \varphi \times \partial_2 \varphi|}$

$$\Rightarrow \int_{\Sigma} F \cdot \hat{n} \, dS = \int_U f \circ \varphi \cdot (\partial_1 \varphi \times \partial_2 \varphi) \, dA \quad \left( \text{if } \hat{n} = + \frac{\partial_1 \varphi \times \partial_2 \varphi}{|\partial_1 \varphi \times \partial_2 \varphi|} \right)$$

Q1:  $U \subseteq \mathbb{R}^2$      $f: U \rightarrow \mathbb{R}$      $C^1$

$$\text{Area}(\Sigma) = \int_U \sqrt{1 + |\nabla f|^2} \, dA$$

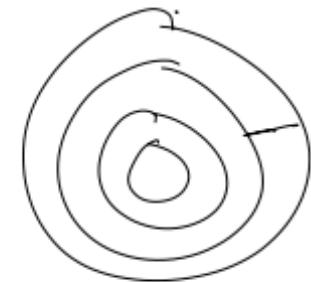
$$\hookrightarrow \text{Area}(\Sigma) = \int_{\Sigma} 1 \, dS$$



Param of  $\Sigma$ : Need  $\varphi: U \rightarrow \Sigma$   
 $x, y \in U$ ,  $\varphi(x, y) = \begin{pmatrix} x \\ y \\ \varphi(x, y) \end{pmatrix} \in \Sigma$

$$\partial_1 \varphi = \begin{pmatrix} 1 \\ 0 \\ \partial_x \end{pmatrix}$$

$$\partial_2 \varphi = \begin{pmatrix} 0 \\ 1 \\ \partial_y \end{pmatrix}$$



adjugate  
adjunct

transpose of cof

$$A = (a_{ij})$$

Q: Formula for  $A^{-1}$ :  $\frac{1}{\det(A)}$

$$\text{adj}(A)$$
$$\boxed{\text{cof}(A)^T}$$

$\text{cof}(A) = \text{cofactor matrix of } A = (c_{ij})$

$\rightarrow c_{ij} = (-1)^{i+j} \det (\text{A with } i^{\text{th}} \text{ row & } j^{\text{th}} \text{ col removed})$

Check  $A \cdot \text{cof}(A)^T = \det(A) I$

$$A^u \times A^v = \text{adj}(A)^T \quad u \times v$$

Stokes Theorem Pf:

$(\Sigma, \hat{n})$  oriented surface  
 $F: \Sigma \rightarrow \mathbb{R}^3$   $C$

$$\oint_{\partial\Sigma} \nabla_x F \cdot d\ell = \iint_{\Sigma} \nabla_x F \cdot \hat{n} \, dS$$

$$\begin{aligned} \int_D F \, dV &= \int_{\partial D} F \cdot \hat{n} \, dS \\ \int_a^b f' \, dx &= f(b) - f(a) \end{aligned}$$

Pf: ① Param  $\Sigma : U \subseteq \mathbb{R}^2$

$$\textcircled{a} \quad \iint_{\Sigma} \nabla_x F \cdot \hat{n} \, dS = \iint_U (\nabla_x F) \circ \varphi \cdot (\partial_1 \varphi \times \partial_2 \varphi) \, dA$$



$$\textcircled{2} \int_{\partial\Omega} F \cdot d\ell = \int_{\partial\Omega} (\nabla \varphi)^T F \circ \varphi \cdot d\ell = \int_{\partial\Omega} G \cdot d\ell$$

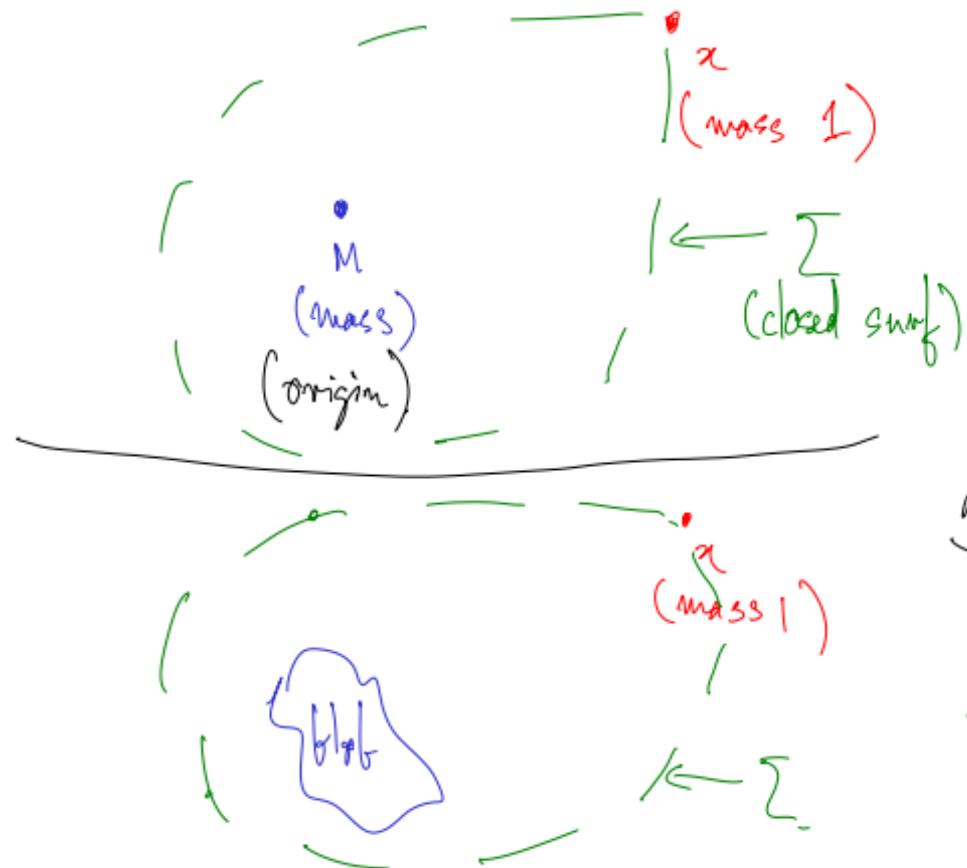
$$\begin{aligned} & \text{Greens theorem} \\ &= \iint_U (\partial_1 G_2 - \partial_2 G_1) \, dA \end{aligned}$$

$$= \iint_U (\nabla \times F \circ \varphi) \cdot (\partial_1 \varphi \times \partial_2 \varphi) \, dA$$

$$= \sum F \cdot \hat{n} \, dS \quad \text{Q.E.D.}$$

$$\boxed{\begin{array}{l} G = (\nabla \varphi)^T F \circ \varphi \\ \partial_1 G_2 - \partial_2 G_1 = (\underbrace{\partial_1 \varphi \times \partial_2 \varphi}_{(\nabla \times F) \circ \varphi}) \cdot (\underbrace{\nabla \times F}_{\nabla \times F}) \end{array}}$$

Hausser law:  $\mathbb{R}^3$



Q:  $g(x)$  = force exerted by a unit mass at  $x$ .

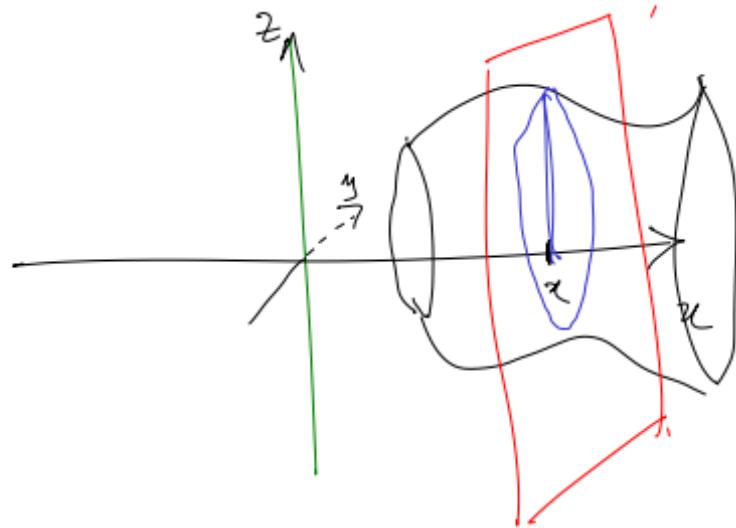
$$g(x) = -G M \left( \frac{x}{|x|^3} \right)$$

$\nabla N$  (Newton potential)

$g(x)$  = force exerted by a unit mass at  $x$

$$\int g(x) \cdot \hat{n} dS = (\text{const}) (\text{mass enclosed by } \Sigma)$$

Q<sup>2</sup>



$$\Sigma = \left\{ y^2 + z^2 = f(x)^2 \right\}$$