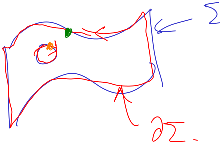
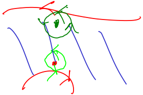
Goal: $\underbrace{\text{Stokes}}_{\text{Pf}}$ + $\underbrace{\text{Div thm}}_{\text{IOU}}$

Statements: Stokes Thm

1. $(\Sigma, \hat{n})$ an oriented surface.
2. Σ is bdd & $\partial\Sigma$ is the finite union of piecewise C^1 curves.
3. Orient $\partial\Sigma$ by traversing it CCW wrt an observer standing with feet on Σ & head pointing towards $\hat{n}$

(Right hand rule: Put right hand on surface with thumb towards $\hat{n}$, the fingers point CCW)





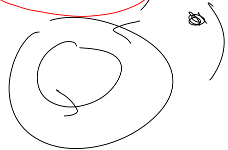
④ Say $F : \overline{\Sigma} \longrightarrow \mathbb{R}^3$ is C^1 $\quad (\overline{\Sigma} = \Sigma \cup \partial\Sigma)$

Thm
$$\int_{\Sigma} (\nabla \times F) \cdot \hat{n}\, dS = \oint_{\partial\Sigma} F \cdot d\ell$$

$$\nabla \times F = \text{curl } F = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

showed up in Greens thm.

Q: What does $\nabla \times F$ mean? ("swirlyness")



(Pf of Stokes thm = Greens thm + change of coordinates)

Divergence Thm: $U \subseteq \mathbb{R}^3$ a domain

1. U is bdd
2. ∂U is a piecewise C^1 surface.
3. Orient ∂U by choosing $\hat{n}$ to be the OUTWARD pointing normal vector.
4. If $F : \overline{U} \to \mathbb{R}^3$ is C^1 $(\overline{U} = U \cup \partial U)$

then

Similar to HW13 4(a). (In 2D reduces to Green's thm).

$$\int_U \underbrace{(\nabla \cdot F)}\ \underbrace{dV}_{\text{vol int}} = \int_{\underbrace{\partial U}_{\text{bdry of } U}} F \cdot \hat{n}\ \underbrace{dS}_{\text{surf int}}$$

Divergence of $F = \nabla \cdot F$
$= \operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$

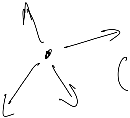
(Pf of div thm = (1) check on a cube (Fubini + FTC) & (2) change coordinates)

(General formulation: Greens thm, Stokes thm & div thm. are all special cases of the general Stokes thm which says

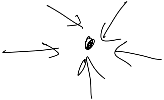
$$\int_M \underbrace{d\omega}_{\text{ext derivative}} = \int_{\partial M} \omega \qquad (\omega \to \text{diff form}) \;)$$

Intuition behind div: measures "compressibility" / sources & sinks

(IOU reason).

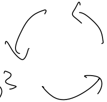


(high +ve div)



(-ve div)

Pf of Stokes thm + curl intuition:



$\nabla \times F$ measures "swirl" (rotation).

Say $u(x)$ = velocity of a fluid (water) at $x \in \mathbb{R}^3$

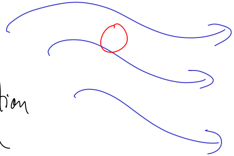
Put a small ~~ball~~ light ball in the fluid.

Ball will follow the fluid.

Ball also starts spinning.

→ axis of rotation is $\nabla \times u$

→ magnitude of ang velocity $\dfrac{|\nabla \times u|}{2}$



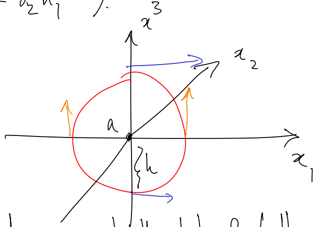
Reason :

$$\nabla \times u = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \boxed{\partial_3 u_1 - \partial_1 u_3} \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

Q : ~~avg~~ speed of rotation of ball in the x_1 x_3 plane.

(claim this is prop to $(\nabla \times u) \cdot e_2$)

What contributes to rotation :



① Diff between ~~hori~~ x_1 componet of u at the top & bottom

② Diff betw x_3 " " " " " right & left.

$\downarrow$ (a) CCW rot from (1) prop to : $u_1(a-he_3) - u_1(a+he_3)$

(b) CCW rot from (2) " " $u_3(a+he_1) - u_3(a-he_1)$

CCW rot prop to $h \underbrace{\left(\frac{u_1(a-he_3) - u_1(a+he_3) + u_3(a+he_1) - u_3(a-he_1)}{h} \right)}_{\lim_{h \to 0}}$

(Ball of rad h rotates at speed prop to $h \cdot |\nabla \times u|$)

$$2 (\underbrace{-\nabla \times u}_{\text{curl } u}) \cdot e_2 = 2 \left(-\partial_3 u_1 + \partial_1 u_3(a) \right)$$

$\nabla \longrightarrow$ grad

$\nabla \times \longrightarrow$ curl

$\nabla \cdot \longrightarrow$ div