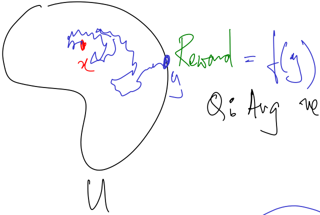
Last (opt prob) on HW: $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$, C^2

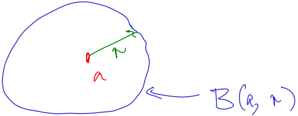
(Harmonic functions) $\Delta u = 0$ $\quad (\Delta u = \partial_1^2 u + \partial_2^2 u + \partial_3^2 u)$

$$\boxed{\forall a \in \mathbb{R}^3, \quad u(a) = \frac{1}{4\pi r^2} \int_{\partial B(a,r)} u \, dS}$$



Q: Avg reward? $\quad u(x) =$ avg reward starting from x

<u>Claim:</u> $\Delta u = 0$ & $u = f$ on ∂U



Surface Int (Vector form)

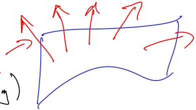
① <u>Oriented Surfaces</u>: We say $(\Sigma, \hat{n})$ is an oriented surface if

(a) $\Sigma \subset \mathbb{R}^3$ is a surface

& (b) $\forall x \in \Sigma$, $\hat{n}(x) \in \mathbb{R}^3$ is a <u>unit</u> normal to Σ

→ (c) $\hat{n} : \Sigma \longrightarrow \mathbb{R}^3$ is $\boxed{\text{cts}}$

u (velocity)

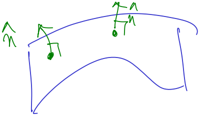


$$\int_\Sigma u \cdot \underbrace{\hat{n}}_{\boxed{\text{unit normal}}} \, dS = \text{fluid flux through } \Sigma$$

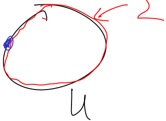
Typical choices for orientation:

① $U \subset \mathbb{R}^3$ an open set

$\Sigma = \partial U$ (bdry of U)



Orientation: Choose the unit normal that points OUTSIDE U.



② Graphs: $U \subseteq \mathbb{R}^2$ & $f : U \to \mathbb{R}$.

Σ = graph of f. $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in U \ \& \ z = f(x,y) \}$.

Orientation: → Choose the unit normal that points UPWARD

$$\hat{n} = \begin{pmatrix} -\partial_x f \\ -\partial_y f \\ +1 \end{pmatrix} \cdot \frac{1}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}$$

in

(formula for normal ~~probability~~)

(3) Surfaces without Orientation : $\longrightarrow$ Klein bottle, $\mathbb{RP}^2$
Mobius strip. (try & prove).

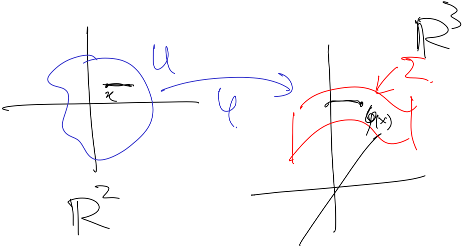
Vector S-Int : $(\Sigma, \hat{n})$ an oriented Surface.
$u : \Sigma \longrightarrow \mathbb{R}^3$ a fn.

$$\int_{\Sigma} u \cdot \hat{n} \, dS \longrightarrow \text{flux through } \Sigma. \to \text{surface int.}$$

Compute : (1) Param Σ. Say $\varphi : U \longrightarrow \Sigma$ is a C^1 param
$(U \subseteq \mathbb{R}^2)$.

① $x \in U$. $\quad a = \varphi(x)$.

$\partial_1 \varphi(x)$ & $\partial_2 \varphi(x)$ are tgt vect to Σ at a.



(Reason: Let $\gamma(t) = \varphi(x + t e_i)$

note $\gamma(t) \in \Sigma \ \forall t \in \mathbb{R}$

$\gamma'(t)$ is tgt to Σ.

Chain rule $\underbrace{\gamma'(0)}_{\text{tgt at } a} = \partial_i \varphi(x)$)

② Normal vector at a: $\quad v = \partial_1 \varphi \times \partial_2 \varphi \quad \Rightarrow v \cdot \partial_1 \varphi = 0$

& $v \cdot \partial_2 \varphi = 0$

($\partial_1 \varphi$ & $\partial_2 \varphi$ are LI $\Rightarrow v$ is a nonzero)

③ Say $(\Sigma, \hat{n})$ is an oriented surface.

$\varphi : U \to \Sigma$ is a param. $x \in U$, $a \in \Sigma$ $a = \varphi(x)$

At a: 2 normal vectors: $\hat{n}$ (from orientation)

or $$\frac{\partial_1 \varphi \times \partial_2 \varphi}{|\partial_1 \varphi \times \partial_2 \varphi|}.$$

Note $\forall a \in \Sigma$, we must have $$\underbrace{\frac{\partial_1 \varphi \times \partial_2 \varphi}{|\partial_1 \varphi \times \partial_2 \varphi|}}_{\text{cts}} = \pm \underbrace{\hat{n}}_{\text{cts}}$$

$\Rightarrow$ must have $\boxed{\dfrac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi\times\partial_2\varphi|} = +\hat{n} \quad \text{on all of } \Sigma}$

OR: $\dfrac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi\times\partial_2\varphi|} = -\hat{n} \quad \text{on all of } \Sigma.$

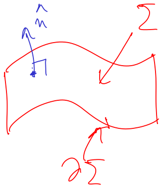
Always prefer param when this holds.

In this case $\displaystyle\int_\Sigma u\circ\hat{n}\, dS = \int_U u\circ\varphi \cdot \underbrace{\hat{n}\circ\varphi}_{\frac{\partial_1\varphi\times\partial_2\varphi}{\cancel{|\partial_1\varphi\times\partial_2\varphi|}}}\ \cancel{|\partial_1\varphi\times\partial_2\varphi|}\, dA$

$$\Rightarrow \int_\Sigma u \cdot \hat{n}\, dS = \int_U (u \circ \varphi) \cdot (\partial_1 \varphi \times \partial_2 \varphi)\, dA$$

Stokes & Div thm:

① $(\Sigma, \hat{n})$ an oriented surface.



$\partial\Sigma \to$ Orient bdry of CCW wrt $\hat{n}$ ~~(ext bdries)~~

~~CW wrt $\hat{n}$ (int bdries)~~

If $F : \Sigma \to \mathbb{R}^3$ is C^1 (& Σ is bdd) then

$$\int_{\Sigma} \underbrace{\nabla \times F}_{\text{curl of } F} \cdot \hat{n}\, dS \overset{\text{stokes thm}}{=\!=} \oint_{\partial \Sigma} F \cdot dl$$

line int

$$\nabla \times F = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

Q: $\Sigma = \{ y^2 + z^2 = 1, \ x \in (-1, 1), \ z > 0 \}$

$F = e_3$ $\qquad \int F \cdot \hat{n} \, dS$

4: Adjugate: $A = \begin{pmatrix} a_{ij} \end{pmatrix}$

Q: Formula for $A^{-1} = \frac{1}{\det A} \overbrace{\left(\text{transpose of cofactor matrix} \right)}^{\operatorname{adj}(A)}$

$C_{ij} = (-1)^{i+j} \det(A \text{ with row } i \text{ \& col } j \text{ removed})$