21-268: Multidimensional Calculus **Spring 2020**

Recitation April 21

Surface Integrals of Scalar Functions

Recall: Let $\rho : \mathbb{R}^3 \to \mathbb{R}$ and $\Sigma \subset \mathbb{R}^3$ is a surface. To compute $\int_\Sigma \rho \, \mathrm{d}S$, we find a parametrization $\varphi : U \to \Sigma$ where $U \subset \mathbb{R}^2$ is a nice (open) set, and then

$$\int_\Sigma \rho \, \mathrm{d}S = \int_U \rho \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, \mathrm{d}A$$

Example 1:

Compute $\int_\Sigma x^2 y + z \, \mathrm{d}S$, where $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$ is the lateral surface of a cylinder.

Solution:

We first find a parametrization $\varphi : U \to \Sigma$, $\varphi : (\theta, h) \mapsto (\cos\theta, \sin\theta, h)$, where $U = \{(\theta, h) \in \mathbb{R}^2 : \theta \in [0, 2\pi), h \in [0, 1]\}$. Thus,

$$\begin{aligned}
\int_\Sigma x^2 y + z \, \mathrm{d}S &= \int_U (\cos^2\theta \sin\theta + h) \, |\partial_1 \varphi \times \partial_2 \varphi| \, \mathrm{d}A \\
&= \int_U (\cos^2\theta \sin\theta + h) \left| \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| \mathrm{d}A \\
&= \int_U (\cos^2\theta \sin\theta + h) \left| \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \right| \mathrm{d}A \\
&= \int_U (\cos^2\theta \sin\theta + h) \, \mathrm{d}A \\
&= \int_0^1 \int_0^{2\pi} \cos^2\theta \sin\theta + h \, \mathrm{d}\theta \, \mathrm{d}h \quad \text{(by Fubini's theorem)} \\
&= \int_0^1 \left(-\frac{1}{3} \cos^3\theta + h\theta \right)_{\theta=0}^{2\pi} \mathrm{d}h \\
&= \int_0^1 2\pi h \, \mathrm{d}h \\
&= 2\pi \left(\frac{1}{2} h^2 \right)_{h=0}^{1} = \pi
\end{aligned}$$

Example 2:

Compute the lateral surface area of a cone with height h and base radius r.

Solution:

To make our computation simpler, we can embed such a cone in $\mathbb{R}^3$ upside down, which is $\{(x, y, z) \in \mathbb{R}^3 : z \in [0, h], \sqrt{x^2 + y^2} \le \frac{r}{h}z\}$. Its lateral surface is $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z \in [0, h], \sqrt{x^2 + y^2} = \frac{r}{h}z\}$.

We find a parametrization $\varphi : U \to \Sigma$, $\varphi : (\theta, t) \mapsto (\frac{rt}{h}\cos\theta, \frac{rt}{h}\sin\theta, t)$, where $U = \{(\theta, t) \in \mathbb{R}^2 : \theta \in [0, 2\pi), t \in [0, h]\}$. Thus, the lateral surface area of the cone is

$$\begin{aligned}
\int_\Sigma 1 \,\mathrm{d}S &= \int_U |\partial_1\varphi \times \partial_2\varphi| \,\mathrm{d}A \\
&= \int_U \left| \begin{pmatrix} -\frac{rt}{h}\sin\theta \\ \frac{rt}{h}\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{r}{t}\cos\theta \\ \frac{r}{h}\sin\theta \\ 1 \end{pmatrix} \right| \mathrm{d}A \\
&= \int_U \left| \begin{pmatrix} \frac{rt}{h}\cos\theta \\ \frac{rt}{h}\sin\theta \\ -\frac{r^2t}{h^2} \end{pmatrix} \right| \mathrm{d}A \\
&= \int_U \frac{rt}{h}\sqrt{1 + \frac{r^2}{h^2}} \,\mathrm{d}A \\
&= \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \int_0^{2\pi} \int_0^h t \,\mathrm{d}t \,\mathrm{d}\theta \quad \text{(by Fubini's theorem)} \\
&= \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \int_0^{2\pi} \frac{1}{2}h^2 \,\mathrm{d}\theta \\
&= \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \cdot \pi h^2 \\
&= \pi r\sqrt{h^2 + r^2}
\end{aligned}$$

A Formula for Surface Integral of Vector-valued Functions

Let $U \subset \mathbb{R}^2, \Sigma \subset \mathbb{R}^3$ be a surface, and $\varphi : U \to \Sigma$ be a parametrization of Σ. We know $\partial_1\varphi(x), \partial_2\varphi(x)$ span the tangent space of Σ at x, so $\partial_1\varphi(x) \times \partial_2\varphi(x)$ is a normal vector of Σ at x.

Now we want to derive a formula for

$$\int_\Sigma u \cdot \hat{n} \, dS$$

Since $\partial_1\varphi \times \partial_2\varphi$ is a normal vector, a unit normal vector is $\dfrac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|}$ Now viewing it as a surface integral of scalar functions, we have

$$\begin{aligned}
\int_\Sigma u \cdot \hat{n}\, dS &= \int_U (u \cdot \hat{n}) \circ \varphi |\partial_1\varphi \times \partial_2\varphi|\, dA \\
&= \int_U (u \circ \varphi) \cdot (\hat{n} \circ \varphi) |\partial_1\varphi \times \partial_2\varphi|\, dA \\
&= \int_U (u \circ \varphi) \cdot \frac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|}\ |\partial_1\varphi \times \partial_2\varphi|\, dS \\
&= \int_U (u \circ \varphi) \cdot (\partial_1\varphi \times \partial_2\varphi)\, dS
\end{aligned}$$

Of course it can also be the negative of this, depending on which orientation one chooses.

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Problem Calculate the flux of $F = (x, y, z^4)$ across the cone given by $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, in the downward direction.

Solution A more mathematical way to say this is: Calculate

$$\int_\Sigma F \cdot \hat{n}\, dS$$

where $\Sigma = \{(x, y, \sqrt{x^2 + y^2}) : x, y \in \mathbb{R}, 0 \leq x^2 + y^2 \leq 1\}$

First we give a easy-to-calculate parametrization of Σ,

$$\varphi : [0, 2\pi] \times [0, 1] \to \Sigma \quad \varphi(r, \theta) = (r\cos\theta, r\sin\theta, r)$$

and calculate $\partial_r\varphi \times \partial_\theta\varphi$.

$$\begin{aligned}
\partial_r\varphi(r, \theta) &= (\cos\theta, \sin\theta, 1) \\
\partial_\theta\varphi(r, \theta) &= (-r\sin\theta, r\cos\theta, 0)
\end{aligned}$$

So

$$\partial_r\varphi(r, \theta) \times \partial_\theta\varphi(r, \theta) = (-r\cos\theta, -r\sin\theta, r)$$

We want the downward direction, so third coordinate should be negative, so we can take $\hat{n} \circ \varphi$ to be $(r\cos\theta, r\sin\theta, -r)/|(r\cos\theta, r\sin\theta, -r)|$

Then

$$\begin{aligned}\int_\Sigma F \cdot \hat{n}\, dS &= \int_0^{2\pi} \int_0^1 F \circ \varphi \cdot (r\cos\theta, r\sin\theta, -r)\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^1 r^2\cos^2\theta + r^2\sin^2\theta - r^5\ dr\ d\theta \\ &= 2\pi \left(\frac{r^3}{3} - \frac{r^6}{6}\right)\Big|_{r=0}^1 = \frac{\pi}{3}\end{aligned}$$