Last time: Surface integrals.

$\underbrace{\Sigma \subset \mathbb{R}^3 \text{ a surface}}_{\substack{\text{plate} \\ \text{(non unif density)}}}, \quad \underbrace{\rho}_{\text{density}} : \Sigma \longrightarrow \mathbb{R}$

$\int_\Sigma \rho \, dS$ = surface int (total mass)

$\lim_{\|P\| \to 0} \sum \text{area}(R_i)\, \rho(\xi_i)$ (last time)

Goal 1: Formula converting surface int into an area int.

① Divide Σ into regions that can be parametrized.

② Say $\Sigma = \bigcup \Sigma_i$ (Σ_i = one of the above regions)

Say $\varphi : U \to \Sigma_i$ is a param of Σ_i

($U \subseteq \mathbb{R}^2$, $\varphi : U \to \Sigma_i$ is C^1, bij & $\forall x \in U$ rank$(D\varphi) = 2$)

Claim : $\underbrace{\int_{\Sigma_i} \rho \, dS}_{\text{surface int}} = \underbrace{\int_U \rho \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA}_{\text{area int.}}$

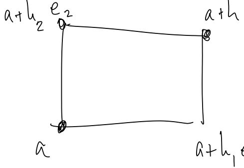
Reason: (1) If $P \subseteq \mathbb{R}^3$ is a parallelogram with sides u & v $(u, v \in \mathbb{R}^3)$

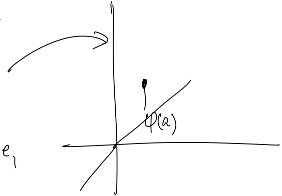
then $\text{area}(P) = |u|\,|v|\,\sin(\theta) = |u \times v|$

(2) Say $R \subseteq \mathbb{R}^2$ is a small rect.

Q: Find $\text{Area}(\varphi(R))$ in terms of φ & $\text{area}(R)$

$\varphi(R) \approx$ a parallelogram vertices
$\varphi(a)$, $\varphi(a+h_1e_1)$, $\varphi(a+h_2e_2)$
$\varphi(a+h)$





$$\varphi(a+h_1e_1) \approx \varphi(a) + D\varphi_a(h_1e_1)$$

$$= \varphi(a) + h_1 \underbrace{D\varphi_a e_1}_{\partial_1\varphi(a)}$$

$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

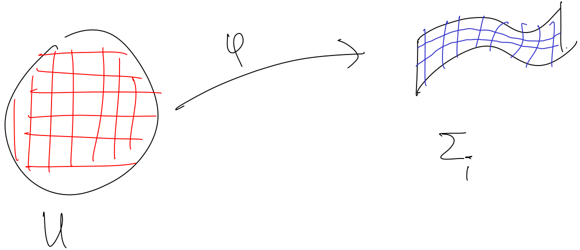
$$\partial_1\varphi = \begin{pmatrix} \partial_1\varphi_1 \\ \partial_1\varphi_2 \\ \partial_1\varphi_3 \end{pmatrix}$$

$$\therefore\ \varphi(a+h_ie_i) \approx \varphi(a) + h_i\,\partial_i\varphi(a)$$

$\Rightarrow \varphi(R) \approx$ a //gm with sides $\approx h_1\partial_1\varphi(a)$ & $h_2\partial_2\varphi(a)$.

$$\Rightarrow \text{area}(\varphi(R)) \approx \left|(h_1\partial_1\varphi(a)) \times h_2\partial_2\varphi(a)\right| = \underbrace{h_1h_2}_{\text{area}(R)}\,|\partial_1\varphi \times \partial_2\varphi|$$

③ $\int$. Int

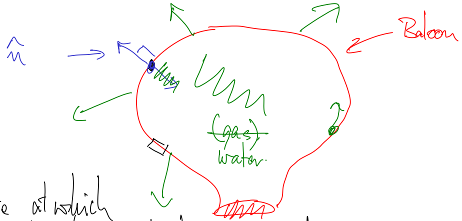


① Partition U into small rect R_i. Let $P_i = \varphi(R_i)$

② $\text{Area}(P_i) \approx \text{area}(R_i)\,|\partial_1\varphi \times \partial_2\varphi|$

③ $$\int_{\Sigma_i} \rho \, dS = \lim_{|P|\to 0} \sum \rho(\xi_i)\,\text{area}(P_i) = \lim_{|P|\to 0} \sum \underbrace{\rho(\xi_i)}_{\rho\circ\varphi}\, \underline{|\partial_1\varphi \times \partial_2\varphi|}\,\text{area}(R_i)$$

$$= \int_U \rho \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA$$



II : Vector Surface integrals.

Intuition : Baloon filled with gas. Gas escapes out.

Q : What is the "flux" (rate at which it flows though the surface).

Let $u(x)$ = velocity of the gas at $x \in \mathbb{R}^3$ ($u(x) \in \mathbb{R}^3$)

Let Σ = surface of the baloon. (say density = 1)

Guess : Flux though the surface : $\boxed{\int_\Sigma u(x) \cdot \hat{n} \, dS}$

(Orientation IOV)

① Specify at every point on the surface a unit normal vector: $\hat{n}$

normal to Σ.

For baloon: Choose $\hat{n}$ to always point outside the baloon

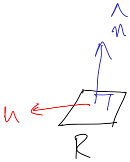
② Let R be a small patch on the baloon.

Fluid flux through R = rate at which fluid flows through R

= (prop to fluid velocity in direction of $\hat{n}$)

(prop to area (R))

= area (R) $u \cdot \hat{n}$



③ Whole surface: Σ over all small patches.

Fluid flux through $\Sigma = \lim\limits_{\|P\| \to 0} \left(\sum \underbrace{u \cdot \hat{n}} \text{ area}(R) \right)$

(whole surface)

component of fluid velocity in direction $\hat{n}$

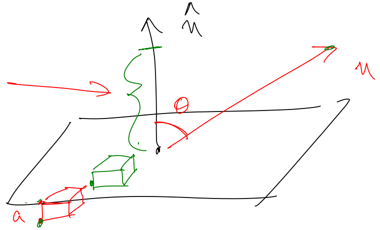
$\Rightarrow$ Fluid flux through $\Sigma = \displaystyle\int_{\Sigma} (u \cdot \underline{\hat{n}}) \underbrace{dS}$

(surface integral)

Q: What is $\hat{n}$ (orientation)

Q: Formula to convert to a std S. int.

$u \cdot \hat{n}$



R

$\boxed{u \cdot \hat{n}}$

Flow for time dt

Cube moves to a cube based at $\boxed{a + \boxed{u\, dt}}$

Q: How much of the cube is outside?

↳ how high is the top of the cube