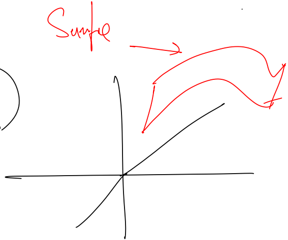
Math: Surface integrals.

$U \subseteq \mathbb{R}^2$ (plate) — non-unif density)

$\rho : U \to \mathbb{R}$ → density

Total mass: Area int: $\int_U \rho \, dA$




$\Sigma \subseteq \mathbb{R}^3$ a surface (soap film non-unif thickness plate)

$\rho : \Sigma \to \mathbb{R}$ (density of the surface)

Q: Total mass: → Surface integral

Notation: $\int_\Sigma \rho \, dS = \int_\Sigma \rho \, d\sigma$

Def: ① Partition the surface into small regions $R_i$

$R_i \approx$ a parallelograms (or rectangles)

② $P = \bigcup R_i$ (a partition)

$\|P\|$ = mesh size = $\max_i$ (largest side length $R_i$)

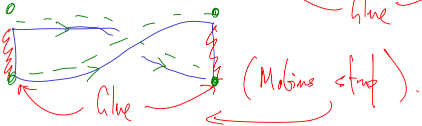
③ Define $$\int_\Sigma \rho \, dS = \lim_{\|P\| \to 0} \underbrace{\sum_i}_{\text{sum}} \text{area}(R_i)\, \rho(\xi_i) \quad (\xi_i \in R_i)$$

Goal 1: Formula to compute $\int_\Sigma \rho \, dS$ as an area integral.

Step 1: Parametrize the surface.

**Def:** $S \subset \mathbb{R}^3$ a surface (piecewise $C^1$ is OK)
$U \subseteq \mathbb{R}^2$ a nice open set.
We say $\varphi : U \to S$ is a ($C^1$) param if
① $\varphi$ is bij & $C^1$ & ② $\forall x \in U$, Rank $(D\varphi_x) = 2$ (full rank).

**Remk:** Every curve can be param, but not every surface can be!
→ Surfaces can be weird:
① Mobius strip

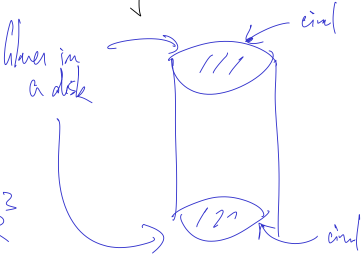
Q: Bdry of a mobius strip? → One (twisted) circle.

↳ Bdry of a cylinder? → Two disj circles.

Bdry of Mobius strip = 1 circle

Glue a disk

Surface that can't be embedded in $\mathbb{R}^3$ (called $\mathbb{R}P^2$)



To compute S-int we divide $\Sigma$ into smaller surfaces each of which can be param.

i.e. Write $\Sigma = \bigcup_1^n \Sigma_i$ & write $\int_\Sigma \rho \, dS = \overset{\text{sum}}{\sum} \int_{\Sigma_i} \rho \, dS$.

(sigma i)

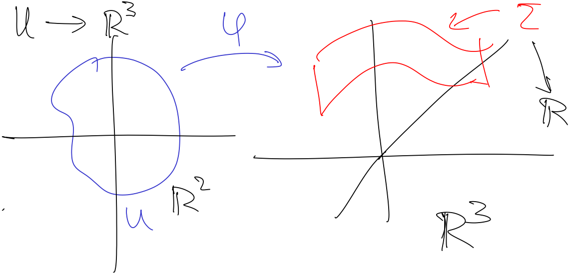
<u>Step 2:</u> Formula for $\int_\Sigma \rho \, dS$ if $\Sigma$ can be param.

Formula: If $\varphi : U \to \Sigma$ is a $C^1$ param then

then $$\int_\Sigma \rho \, dS = \iint_U (\rho \circ \varphi) \, \Big| \underbrace{\partial_1 \varphi}_{\in \mathbb{R}^3} \times \underbrace{\partial_2 \varphi}_{\in \mathbb{R}^3} \Big| \, dA$$

① $\partial_i \varphi = \begin{pmatrix} \partial_i \varphi_1 \\ \partial_i \varphi_2 \\ \partial_i \varphi_3 \end{pmatrix}$ $\qquad \partial_i \varphi : U \to \mathbb{R}^3$

② $\partial_1 \varphi \times \partial_2 \varphi$ = cross prod of $\partial_1 \varphi$ & $\partial_2 \varphi$.




Why does this work: Intuition → ① Say $P \subseteq \mathbb{R}^3$ is a parallelogram with sides $u$ & $v$. ($u, v \in \mathbb{R}^3$)

What is area $(P)$?

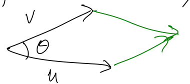
Claim: area$(P)$ $= |u \times v| = |u|\,|v| \sin\theta$

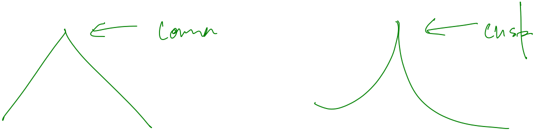
Quick reminder $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$u \times v = \begin{pmatrix} \boxed{u_2 v_3 - u_3 v_2} \\ \boxed{u_3 v_1 - u_1 v_3} \\ \boxed{u_1 v_2 - u_2 v_1} \end{pmatrix} \qquad \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$$

Proof: ① $(u \times v)$ is $\perp$ to both $u$ & $v$.

$\rightarrow$ ② $|u \times v| = |u|\,|v| \sin\theta$
(angle between)

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