Last time:

HW Q 3b: ~~to~~ find Winding # of a circle

(a) center 0 radius 1 (easy)

(b) center ~~(2, 3)~~ (0, 2) radius 1 ⟶ Option 1: parametrize & SUFFER

Option 2: Use greens thm:

Compute $\frac{1}{2\pi} \oint_\Gamma \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} = \frac{1}{2\pi} \oint_\Gamma P\,dx + Q\,dy$

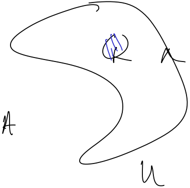
$= \int_{\text{region encl by } \Gamma} \underbrace{(\partial_x Q - \partial_y P)}_{0}\, dx\, dy = 0$

Q: Does this work for (a) !! NO: P & Q are not defined at 0

Today: Finish Pf of Greens thm:

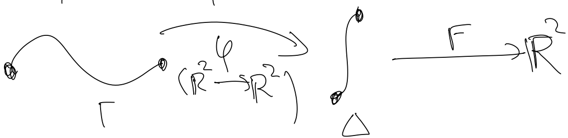
$U \subseteq \mathbb{R}^2$ bdd. $\partial U \to$ piecewise C^1 → Ext bdry CCW, Int bdry CW

$F : \bar{U} \to \mathbb{R}^2$, C^1. Thm: $\displaystyle\oint_{\partial U} F \cdot dl = \int_U (\partial_1 F_2 - \partial_2 F_1)\, dA$



Last time (1) If U = unit square $\Rightarrow$ Greens thm is true

(2) Coordinate change



Then $\int_\Delta F \cdot dl = \int_\Gamma (D\varphi)^T F\circ\varphi \cdot dl$

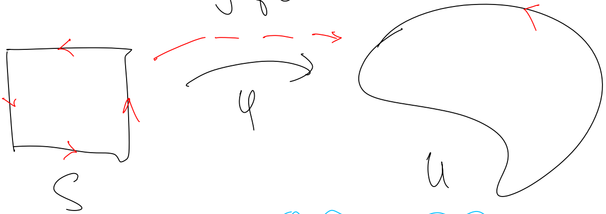
($\Gamma = \varphi^{-1}(\Delta)$ & φ is a coordinate change fn)

(Note: Area int: $\int_V \rho \, dA = \int_U \rho\circ\varphi \, |\det D\varphi| \, dA$

Pf of Greens thm Part 2: (Already know Greens thm in a Sq)

Suppose $\exists\, \varphi : S \longrightarrow U$ which is a C^2 coordinate change fn.
(unit sq)

such that $\underset{\text{on bdry of } S}{\varphi : \partial S} \longrightarrow \partial U$ is bij & preserves orientations.



(in this case can check $\det(D\varphi) > 0$)

Note $\displaystyle\int_{\partial U} F \cdot dl \underset{\substack{\text{coordinate chge}\\ \text{line int}}}{=} \int_{\partial S} \underbrace{(D\varphi)^T F \circ \varphi}_{G} \cdot dl$

Let $G = (D\varphi)^T F \circ \varphi$

$$\Rightarrow \oint_{\partial U} F \cdot dl \overset{\text{c. change}}{=} \oint_{\partial S} G \cdot dl \xrightarrow[\text{sq}]{\text{Greens thm on}} \int_S (\partial_1 G_2 - \partial_2 G_1)\, dA$$

Compute $\partial_1 G_2 - \partial_2 G_1$, where $\begin{pmatrix} G_i \end{pmatrix} = (\partial_j \varphi_i)^T \cdot \begin{pmatrix} F_i \circ \varphi \end{pmatrix}$

$$\boxed{\Rightarrow G_i = \sum_{j=1}^{2} \partial_i \varphi_j \, F_j \circ \varphi}$$

Compute $\partial_1 G_2 - \partial_2 G_1 \longleftrightarrow$ product rule + chain rule

$(*)$ $\boxed{\text{Get } \partial_1 G_2 - \partial_2 G_1 = (\partial_1 F_2 - \partial_2 F_1) \circ \varphi \, (\det D\varphi)}$

Assume (*) Prove Greens thm:

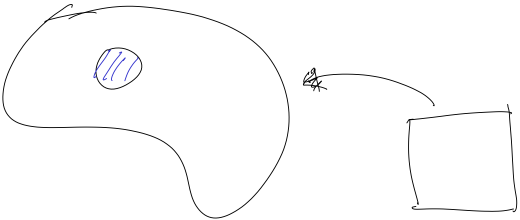
$$\Rightarrow \oint_{\partial U} F \cdot dl \overset{\text{c.c}}{=} \oint_{\partial S} G \cdot dl \xrightarrow{\text{Greens thm on sq}} \int_S \underbrace{(\partial_1 G_2 - \partial_2 G_1)}_{(*)} \, dA$$

$$\int_S (\partial_1 F_2 - \partial_2 F_1) \circ \varphi \; \left| \underbrace{(\det D\varphi)}_{>0} \right| dA \overset{\text{coordinate change}}{=} \int_U (\partial_1 F_2 - \partial_2 F_1) \, dA$$

QED !!

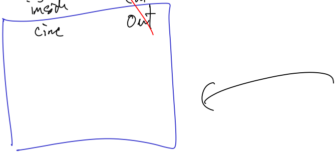
Q: Why did I need φ to be C^2? Is C^1 enough?

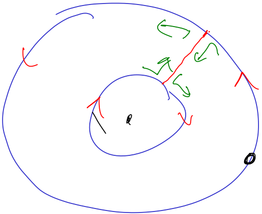
Q2: What if $U =$



Q: Does such a φ exist??

$$\int_{\text{Out circl}} + \cancel{\int_{\text{cut in}}} + \int_{\text{inside circ}} + \cancel{\int_{\text{cut out}}}$$





(Once red cut twice! When we do line integrals this won't matter as we traverse in opp dir!)

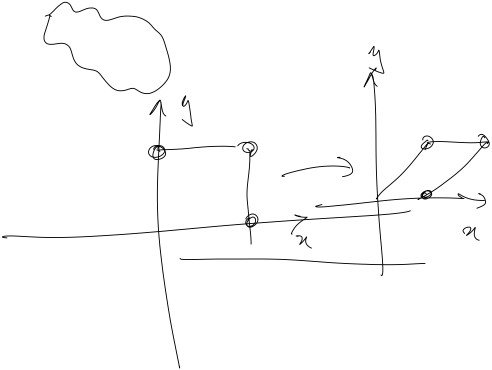
$$\int_{\partial U} (\ ) = \int_U -$$

$$\rightarrow \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

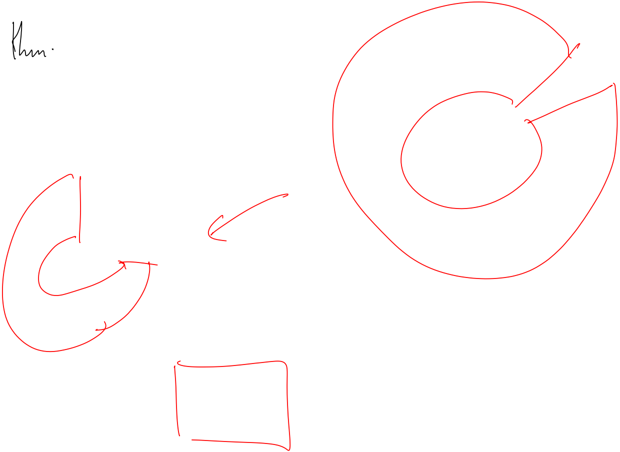
Shear : $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Stretch $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$

Flip $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



3(b) Find them.



⊗ (3b) via fundamental thm.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

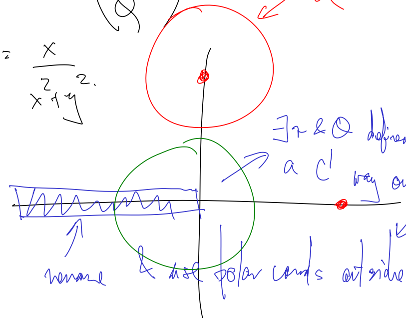
Q: does $\exists\, F$ s.t. $\nabla F = \begin{pmatrix} P \\ Q \end{pmatrix}$

Compute $\nabla\theta$ by diff impl.

$$P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}.$$

$$F = \tan^{-1}\left(\frac{y}{x}\right)$$

θ in polar coordinates



$$\boxed{\oint \underset{\uparrow}{\underset{\text{curv}}{K}} \; |d\ell|}$$

normal curv.

curl only makes sense for vector fctions.

= winding # (something related to μ)

$$3D: \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \qquad \nabla \times \varphi = \begin{pmatrix} \partial_2 \varphi_3 - \partial_3 \varphi_2 \\ -(\partial_1 \varphi_3 - \partial_3 \varphi_1) \\ \boxed{\partial_1 \varphi_2 - \partial_2 \varphi_1} \end{pmatrix}$$

$\boxed{2D: \text{curl}}$: $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Think of

$$\boxed{\nabla \times \omega = \begin{pmatrix} \partial_2 \omega \\ -\partial_1 \omega \end{pmatrix}}$$

$$\varphi(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ 0 \\ \omega(x_1, x_2) \end{pmatrix} : \nabla \times \varphi = \begin{pmatrix} \partial_2 \omega \\ -\partial_1 \omega \\ 0 \end{pmatrix}$$