21-268: Multidimensional Calculus **Spring 2020**

Recitation April 14

The Area of an Ellipse

Green's theorem is handy to transform a hard-to-compute line integral to an easier area integral, and sometimes the other way around.

Example Consider an ellipse in $\mathbb{R}^2$, namely

$$\Gamma := \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$

where $0 < a, b \in \mathbb{R}$. Calculate the area of the region Ω enclosed by this curve Γ.

Solution One way to do it is a simple coordinate change we saw last time.

The other way is through Green's theorem. Recall in the Greens theorem we have

$$\int_\Omega \partial_1 F_2 - \partial_2 F_1 dA = \int_\Gamma F \cdot dl$$

So to calculate

$$\text{Area}(\Omega) = \int_\Omega 1 dA$$

We want to use some F such that $\partial_1 F_2 - \partial_2 F_1 = 1$. There are many choices of F that is valid, but they may lead to different level of difficulty on evaluating the line integral.

Let's first parameterize Γ. Similar to a circle, which is a special case of the an ellipse, we can use $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined via

$$\gamma(t) = (a \cos t, b \sin t)$$

Then if we choose $F = (0, x)$, the line integral becomes

$$\int_\Gamma \begin{pmatrix} 0 \\ x \end{pmatrix} \cdot dl = \int_0^{2\pi} \begin{pmatrix} 0 \\ a \cos t \end{pmatrix} \cdot \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix} dt = ab \int_0^{2\pi} \cos^2 t dt$$

We can also choose $F = (-y/2, x/2)$, than the line integral becomes

$$\frac{1}{2} \int_\Gamma \begin{pmatrix} -y \\ x \end{pmatrix} \cdot dl = \frac{1}{2} \int_0^{2\pi} \begin{pmatrix} -b \sin t \\ a \cos t \end{pmatrix} \cdot \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix} dt = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab$$

(It's not difficult to calculate integration of $\cos^2 t$, but this choice of F makes it even easier)

Another example of Green's theorem

Let U be the region bounded by the functions $y = 2x$ and $y = x^2$ in the first quadrant, and let $F(x, y) = (x^2y, xy^2)$. Compute $\int_{\partial U} F \cdot \mathrm{d}l$ both directly and using Green's theorem.

Solution 1

Let $\gamma : [0, 4] \to \partial U$ be defined as

$$\gamma(t) = \begin{cases} (t, t^2) & \text{if } t \in [0, 2] \\ (4 - t, 8 - 2t) & \text{if } t \in (2, 4] \end{cases}$$

Note that γ is a parametrization of ∂U. Thus,

$$\begin{aligned} \int_{\partial U} F \cdot \mathrm{d}l &= \int_0^4 F \circ \gamma(t) \cdot \gamma'(t)\,\mathrm{d}t \\ &= \int_0^2 \begin{pmatrix} t^4 \\ t^5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} \mathrm{d}t + \int_2^4 \begin{pmatrix} (4-t)^2(8-2t) \\ (4-t)(8-2t)^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \end{pmatrix} \mathrm{d}t \\ &= \int_0^2 t^4 + 2t^7\,\mathrm{d}t + \int_2^4 10(t-4)^3\,\mathrm{d}t \\ &= \left(\frac{1}{5}t^5 + \frac{2}{7}t^6\right)_{t=0}^{2} + \left(\frac{5}{2}(t-4)^4\right)_{t=2}^{4} \\ &= \frac{32}{5} + \frac{256}{7} - 40 = \frac{104}{35} \end{aligned}$$

Solution 2

By Green's theorem,

$$\begin{aligned} \int_{\partial U} F \cdot \mathrm{d}l &= \int_U \partial_1 F_2 - \partial_2 F_1\,\mathrm{d}A \\ &= \int_U y^2 - x^2\,\mathrm{d}A \\ &= \int_0^2 \int_{x^2}^{2x} y^2 - x^2\,\mathrm{d}y\,\mathrm{d}x \quad \text{(by Fubini's theorem)} \\ &= \int_0^2 \left(\frac{1}{3}y^3 - x^2y\right)_{y=x^2}^{2x} \mathrm{d}x \\ &= \int_0^2 \frac{2}{3}x^3 - \frac{1}{3}x^6 + x^4\,\mathrm{d}x \\ &= \left(\frac{1}{6}x^4 - \frac{1}{21}x^7 + \frac{1}{5}x^5\right)_{x=0}^{2} \\ &= \frac{8}{3} - \frac{128}{21} + \frac{32}{5} = \frac{312}{105} = \frac{104}{35} \end{aligned}$$

The Insider

A common place to get confused is when you have a region bounded by two curves, one inside and one outside. The key point to remeber is that the outside curve goes counter-clockwise, but the inside curve goes clockwise.

Let Ω be the region with inside boundary Γ_1 and outside boundary Γ_2. A way to think about it is that if we want to fill the hole, you fill in a region Γ' with Γ_1 as the outside boundary. Then

$$\int_{\Omega+\Omega'} \partial_1 F_2 - \partial_2 F_1 dA = \int_\Gamma F \cdot dl = \left(\int_\Gamma F \cdot dl - \int_{\Gamma'} F \cdot dl\right) + \left(\int_{\Gamma'} F \cdot dl\right)$$

where the line integration over Γ' is counter-clockwise and the two parenthesis corresponds to $\int_\Omega \partial_1 F_2 - \partial_2 F_1 dA$ and $\int_{\Omega'} \partial_1 F_2 - \partial_2 F_1 dA$.

Now we look at an example.

Example Evaluate

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}} dA$$

where $\Omega = \{1 \le x^2 + y^2 \le 4\}$

Solution

Let $F = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} -y \\ x \end{pmatrix}$, then

$$\partial_1 F_2 - \partial_2 F_1 = \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}}$$

So

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}} dA = \int_{\Gamma_2} F \cdot dl - \int_{\Gamma_1} F \cdot dl$$

where $\Gamma_1 = \{x^2 + y^2 = 1\}, \Gamma_2 = \{x^2 + y^2 = 4\}$ and both integrals are evaluated counter-clockwise. Use the parametrization $\gamma_2(t) = (2\cos t, 2\sin t)$, $t \in [0, 2\pi]$, we get

$$\begin{aligned}
\int_{\Gamma_2} F \cdot dl &= \int_0^{2\pi} \begin{pmatrix} -(2\sin t)/2 \\ (2\cos t)/2 \end{pmatrix} \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt \\
&= \int_0^{2\pi} 2\sin^2 t + 2\cos^2 t \; dt \\
&= \int_0^{2\pi} 2 \; dt = 4\pi
\end{aligned}$$

Similarly use the parametrization $\gamma_1(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we get

$$\begin{aligned}\int_{\Gamma_1} F \cdot dl &= \int_0^{2\pi} \begin{pmatrix} -(\sin t)/1 \\ (\cos t)/1 \end{pmatrix} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t \; dt \\ &= \int_0^{2\pi} 1 \; dt = 2\pi\end{aligned}$$

So

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}} dA = 4\pi - 2\pi = 2\pi$$

We can also verify it using polar coordinate,

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}} dA = \int_0^{2\pi} \int_1^2 \frac{1}{r} r \; dr \; d\theta = 2\pi$$

Example when Green's theorem fails

Green's theorem may fail if either the region U is unbounded (example below) or F is not C^1 in U (winding number).

Example

Let $U = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Note that ∂U is the x-axis, which can be parametrized by $\gamma(t) = (t, 0)$ for $t \in \mathbb{R}$. Let $F(x, y) = (xe^{-x^2/2}, x)$. We compute

$$\begin{aligned}\int_{\partial U} F \cdot \mathrm{d}l &= \int_{-\infty}^{\infty} F \circ \gamma(t) \cdot \gamma'(t) \, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} te^{-t^2/2} \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} te^{-t^2/2} \, \mathrm{d}t \\ &= \left(-e^{-t^2/2}\right)_{t=-\infty}^{\infty} \\ &= 0\end{aligned}$$

and

$$\int_U \partial_1 F_2 - \partial_2 F_1 \, \mathrm{d}A = \int_U 1 \, \mathrm{d}A = \infty$$

In this case,

$$\int_{\partial U} F \cdot \mathrm{d}l \neq \int_U \partial_1 F_2 - \partial_2 F_1 \, \mathrm{d}A$$