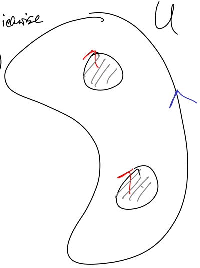
Greens theorem: $U \subseteq \mathbb{R}^2$ a $\boxed{\text{bounded}}$ domain

∂U = boundary of U → (assume this is the finite union of piecewise C^1 curves)

① Exterior bdry is oriented counter clockwise

② All int boundaries oriented clockwise



Let $F : \overline{U} \to \mathbb{R}^2$ be C^1 $\quad (\overline{U} = U \cup \partial U)$

Then

$$\oint_{\underbrace{\partial U}_{\text{bdry of } U}} F \cdot \underbrace{d\ell}_{\text{line int}} = \int_{\underbrace{U}_{\text{domain } U}} (\partial_1 F_2 - \partial_2 F_1)\, dA$$

Alt notation: $P, Q : \overline{U} \to \mathbb{R}$ are 2 C^1 functions

$$\int_{\partial U} (P\,dx + Q\,dy) = \int_U (\partial_x Q - \partial_y P)\, dx\, dy$$

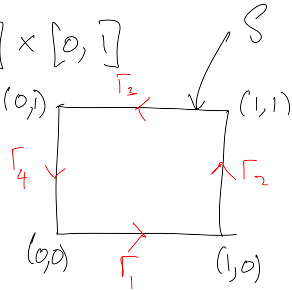
Today Pf of Green's thm:

Part I: Assume U is a Square.

Part II: Get general domains using coordinate changes

(1) Part I: $S = [0,1]^2 = [0,1] \times [0,1]$

$\partial S = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$



Compute $\int_S (\partial_1 F_2 - \partial_2 F_1)\, dA = \int_S \partial_1 F_2 \, dA - \int_S \partial_2 F_1 \, dA$

$$= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \partial_1 F_2(x_1,x_2)\, dx_2\, dx_1 - \int_{x_1=0}^{1} \underbrace{\int_{x_2=0}^{1} \partial_2 F_1(x_1,x_2) \quad dx_2}_{FTC} \quad dx_1$$

$$= \int_{x_2=0}^{1} \Bigg(\underbrace{\int_{x_1=0}^{1} \partial_1 F_2 \quad dx_1}_{FTC} \Bigg) dx_2 - \int_{x_1=0}^{1} \Big(F_1(x_1,1) - F_1(x_1,0) \Big) dx_1$$

$$= \int_{x_2=0}^{1} \Big(F_2(1,x_2) - F_2(0,x_2) \Big) dx_2 - \int_{x_1=0}^{1} \Big(F_1(x_1,1) - F_1(x_1,0) \Big) dx_1$$

$$= \underbrace{\int_{x_2=0}^{1} F_2(1, x_2)\, dx_2}_{\substack{I_2 \\ \text{(right)}}} - \underbrace{\int_{x_2=0}^{1} F_2(0, x_2)\, dx_2}_{\substack{I_4 \\ \text{(left)}}} - \underbrace{\int_{x_1=0}^{1} F_1(x_1, 1)\, dx_1}_{\substack{I_3 \\ \text{(top)}}} + \underbrace{\int_{x_1=0}^{1} F_1(x_1, 0)\, dx_1}_{\substack{I_1 \\ \text{(bottom)}}}$$

$$\Rightarrow \int_S (\partial_1 F_2 - \partial_2 F_1)\, dA = I_1 + I_2 + I_3 + I_4$$

<u>Claim</u>: $\int_{\Gamma_i} F \cdot d\ell = I_i$ (Note Claim $\Rightarrow$ QED (Part I))

Pf of claim: Check $I_3 = -\int_{x_1=0}^{1} F_1(x_1, 1)\, dx_1 = \int_{\Gamma_3} F \cdot d\ell$.

Note $\int_{\Gamma_3} F \cdot dl$: ① param Γ_3

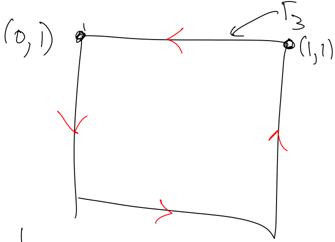
Let $\gamma_3(t) = \begin{pmatrix} 1-t \\ 1 \end{pmatrix}$

$$\Rightarrow \int_{\Gamma_3} F \cdot dl = \int_{t=0}^{1} \begin{pmatrix} F_1 \circ \gamma_3(t) \\ F_2 \circ \gamma_3(t) \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\gamma_3'(t)} dt$$

$$= \int_{t=0}^{1} F_1(1-t, 1)\,(-1)\,dt \overset{\text{change of var}}{=} -\int_{x_1=0}^{1} F_1(x_1, 1)\,dx_1 = I_3$$

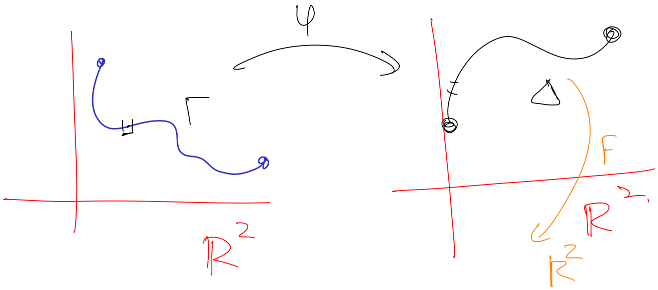
QED.

(Put $x_1 = 1-t$ $\quad dx_1 = -dt$)



Part II of proof: (Use coordinate changes)

Step 1: Figure out a formula for coordinate changes of line integrals.



$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is C^1

$\Gamma \subseteq \mathbb{R}^2$ a curve.

$\Delta = \varphi(\Gamma)$ a curve

$F : \Delta \longrightarrow \mathbb{R}^2$ a C^1 fn.

Goal: $\int_\Delta F \cdot d\ell$ relate it to $\int_\Gamma (F\circ\varphi)\cdot d\ell$

Guess: $\int_\Delta F\cdot d\ell = \int_\Gamma |\det D\varphi|\, F\circ\varphi \cdot d\ell$

$\int$

$\int_\Gamma F\circ\varphi \cdot D\varphi\,(d\ell)$

$\int_\Gamma \left[(D\varphi)^T F\circ\varphi\right]\cdot d\ell$

$(A^T u)\cdot v$

$\|$

$(A^T u)^T v$

$\|$

$u\cdot Av =$

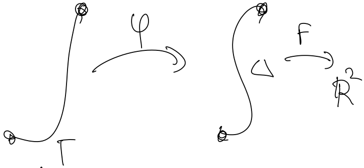
$u^T(Av) = (u^T A)\cdot v$

$(A^T u)^T = u^T A$

$\boxed{\Rightarrow u\cdot Av = (A^T u)\cdot v}$

Claim: $\displaystyle\int_\Delta F \cdot d\ell = \int_\Gamma \left[(D\varphi)^T F\circ\varphi\right]\cdot d\ell$

Pf of claim: Recall $u\cdot(Av) = (A^T u)\cdot v$



① Let $\gamma : [0,1] \to \Gamma$ be a param of Γ

Then $\varphi\circ\gamma$ is a param of Δ. Let $\delta = \varphi\circ\gamma$.

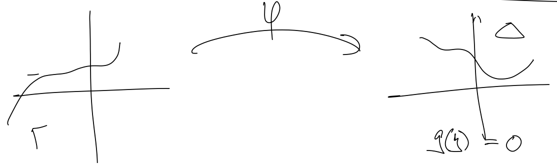
$$\Rightarrow \int_\Delta F\cdot d\ell = \int_0^1 (F\circ\delta(t))\cdot\delta'(t)\,dt = \int_0^1 (F\circ\varphi\circ\gamma(t))\cdot\left(D\varphi_{\gamma(t)}\,\gamma'(t)\right)dt$$

$$= \int_0^1 \left[D\varphi_{\gamma(t)}^T \, F \circ \varphi \circ \gamma(t) \right] \cdot \gamma'(t) \, dt$$

$$= \int_\Gamma \left(D\varphi^T \, F \circ \varphi \right) \cdot dl \qquad \text{QED.}$$

Say $\Delta = \{ g = 0 \}$

$\Gamma = \{ f = 0 \}$



$T = \varphi(O)$

$$\boxed{g = f \circ \varphi}$$

Tgt space of $T = \ker(Df)$

Tgt spce of $O = \ker(Dg) = \ker(D(f \circ \varphi)) = \ker(\underbrace{Df_{\varphi}}_{\text{Tgt vctrs of } T} \cdot \underbrace{D\varphi}) $

$\varphi(x) = Tx$ (T is a matrix)

$D\varphi_x = T$

$f = -\underbrace{\nabla V} \rightarrow V =$ potential