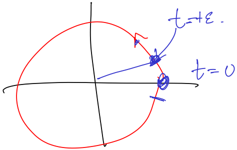
Eg: Winding # along a circle CW & CCW.

$\Gamma = S^1 = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}.$



① $W(\Gamma)$ traversed CCW

$\mapsto \frac{1}{2\pi} \oint_\Gamma \frac{-x_2 \, dx_1 + x_1 \, dx_2}{|x|^2}$

ⓐ $\longrightarrow$ param $S^1 \xrightarrow{ccw} x(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $(t \in (0, 2\pi))$

ⓑ $W(\Gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|x(t)|^2} \left(-\sin t \, (-\sin t) \, dt + \cos t \, (\cos t) \, dt \right)$

$x_1 = \cos t$

$dx_1 = -\sin t \, dt$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1 \, dt = +1.$$

(2) $W(\Gamma)$ traversed CW. (Ans $= -1$)

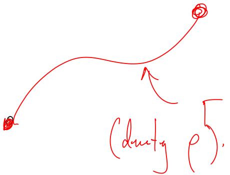
(a) param : $x(t) = (\cos(2\pi - t), \sin(2\pi - t))$ ✓

alternate choice $x(t) = (\cos t, -\sin t)$ ✓

$$W(\Gamma) = \frac{1}{2\pi} \oint_\Gamma \frac{-x_2\, dx_1 + x_1\, dx_2}{|x|^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1}\Big[\big(+\sin t\,(-\sin t\, dt)\big) + \cos t\,(-\cos t)\,dt\Big]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} -1 \, dt = -1.$$

Q1: Center of mass of the wire $= (a, b)$



$a =$

$b =$

For a disk: $a = \frac{1}{M} \int_D x\, \rho(x,y)\, dA \qquad \left(M = \int_D \rho(x,y)\, dA \right)$

For wire: $a = \frac{1}{M} \int_\Gamma x\, \rho(x,y)\, |d\ell| \qquad \left(M = \int_\Gamma \rho(x,y)\, |d\ell| \right)$

Compute: ① param Γ: Let γ be the param.

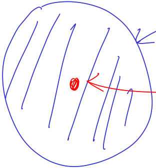
$\hookrightarrow \gamma(t) = (\gamma_1(t), \gamma_2(t))$

$$\int_\Gamma x\, \rho(x,y)\, \underbrace{|dl|} = \int \underline{\gamma_1(t)\, \rho\big(\gamma_1(t), \gamma_2(t)\big)\, |\gamma'(t)|}\, dt$$

line int $\quad dl \longrightarrow \gamma'(t)\, dt$

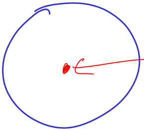
arc len int $\quad |dl| \longrightarrow |\gamma'(t)|\, dt$

$$\gamma'(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} \qquad |\gamma'(t)| = \sqrt{(\gamma_1')^2 + (\gamma_2')^2}$$



Disk of unif density

dut of mass

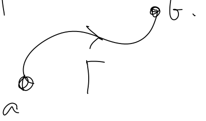


Wine of uniform density

center of mass.

2D: Compute line int via fundamental thm.

① $\int_\Gamma F\cdot dl$: If $F = \nabla\varphi \Rightarrow \int_\Gamma F\cdot dl = \varphi(b)-\varphi(a)$



② To use: Need to know $\exists\, \varphi \ni F = \nabla\varphi$

Easy way to check:

If $F = \nabla\varphi = \begin{pmatrix}\partial_1\varphi \\ \partial_2\varphi\end{pmatrix}$

then $\underbrace{\partial_1 F_2 - \partial_2 F_1} = \partial_1\partial_2\varphi - \partial_2\partial_1\varphi = 0$ if φ is C^2

(Thm: If $\partial_1 F_2 - \partial_2 F_1 = 0$ then $\exists\, \varphi \ni F = \nabla\varphi$ (IOO Pf))

To compute the line int $\to$ still have to find φ (not easy)

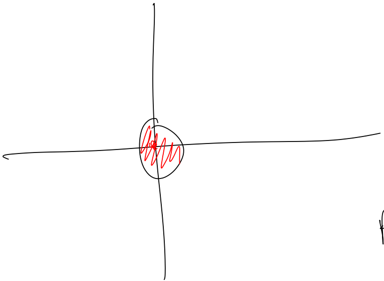
③ One tip: Compute $\boxed{\nabla|x| = \frac{x}{|x|}}$ $(x \in \mathbb{R}^d)$

forces of the form $F(x) = g(|x|) \cdot \frac{x}{|x|}$ are always potential forces.

Claim: $\varphi(x) = G(|x|)$ is the potential $\left(G = \int g\right)$

$\nabla\varphi \overset{\text{chain rule}}{=} G'(|x|)\,\frac{x}{|x|} = g(|x|)\,\frac{x}{|x|}$.

Eg: Gravitational force:



Mass of size M at origin.

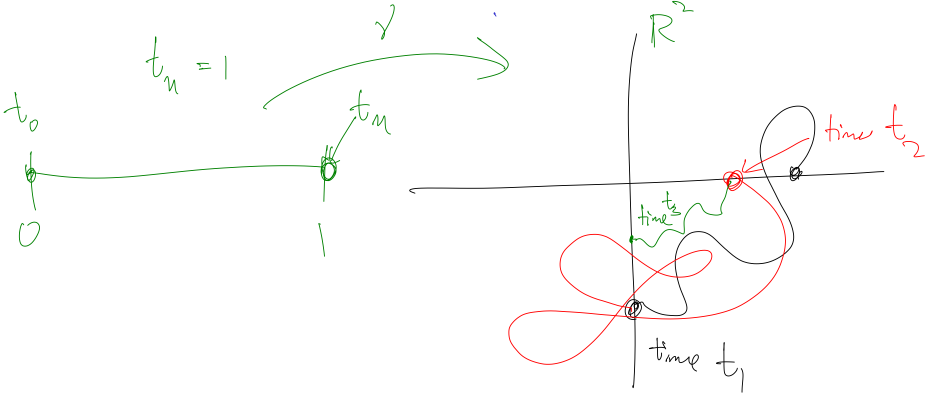
Body of mass m at $x \in \mathbb{R}^3$

Gravitational force: $\frac{G\,Mm}{|x|^2}$ — mag of force.

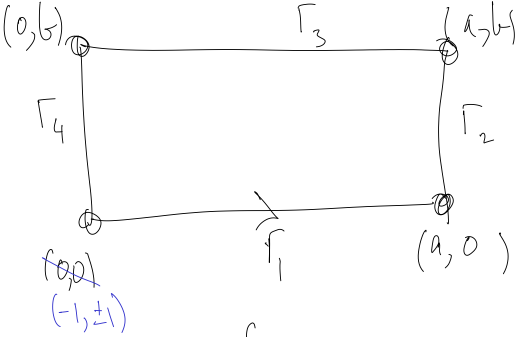
Grav const

dir: radially inward.

$$F(x) = \underbrace{-}_{\text{radially inward.}} \underbrace{\frac{G\,Mm}{|x|^2}}_{\text{mag}} \underbrace{\frac{x}{|x|}}_{\text{unit vector (radially outward)}}$$



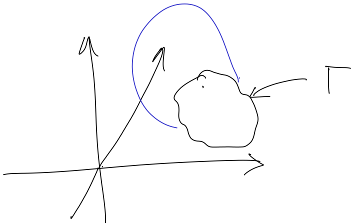
$t_n = 1$
γ
$\mathbb{R}^2$
t_0
t_n
0
1
time t_2
time t_3
time t_1



$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

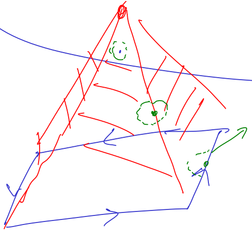
$$\int_{\Gamma} F \cdot dl = \sum_{i=1}^{4} \int_{\Gamma_i} F \cdot dl$$

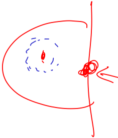
$$\oint B \cdot dl$$



$(-1, \pm 1, -1) \qquad (1, \pm 1, 1)$

Order:
$$\begin{tikzcd} (-1,-1,-1) \arrow[r] & (1,-1,1) \arrow[r] & (-1,+1,-1) \arrow[d] \\ & & (1,+1,1) \arrow[llu, bend left] \end{tikzcd}$$





boiling pt