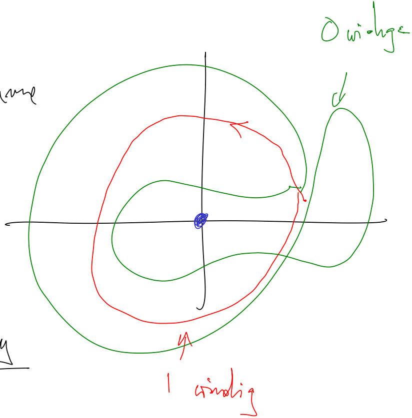
HW Q4 (Winding #)

$\Gamma \subset \mathbb{R}^2$ is a closed curve
(Γ doesn't pass through the origin)

Q: How many times does Γ wind around the origin?

Formula: $W(\Gamma) =$ on HW

$$= \frac{1}{2\pi} \oint_\Gamma \frac{-y\,dx + x\,dy}{x^2+y^2}$$



$$= \frac{1}{2\pi} \oint \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ +x \end{pmatrix} \cdot dl$$

(Intuition:
$$\frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} = \nabla \underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{\text{"}\theta\text{"}}$$
(angle var in Polar))

Wrong Pf that $W(\Gamma)$ = Winding #.

$$W(\Gamma) = \frac{1}{2\pi}\oint_\Gamma \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \cdot dl = \frac{1}{2\pi}\oint_\Gamma \nabla \underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{\theta} \cdot dl$$

$$= \frac{1}{2\pi}\int_\Gamma \nabla\theta \cdot dl = \frac{1}{2\pi}\left(\theta(\text{end pt}) - \theta(\text{start pt})\right)$$

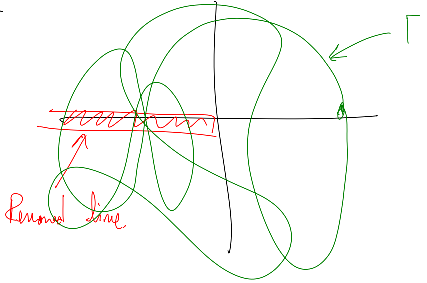
$$= \frac{1}{2\pi}\left(\text{total angle swept out by } \Gamma\right)$$

$$= \# \text{ windings!}$$

QED

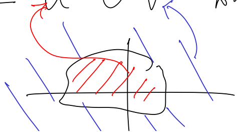
(Error: angle can only be defined if we remove a ~~line~~ ray from the plane

Any curve that winds around the origin will necessarily cross the removed line.
$\Rightarrow$ can't apply fund thm directly.



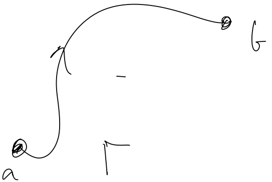
Thm (Jordan curve thm)
(curve with no self intersections)
If $\Gamma \subseteq \mathbb{R}^2$ is a simple closed curve

Then $\mathbb{R}^2 - \Gamma = U \cup V$ where U & V are disj non-empty open sets.



Last time: Fund thm of line integrals.

$$\int_\Gamma \nabla\varphi \cdot dl = \varphi(b) - \varphi(a)$$



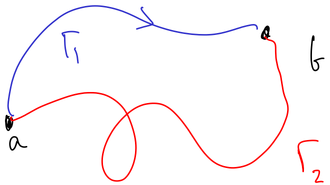
(Last time: $\int_\Gamma \underbrace{-\nabla V}_{F} \cdot dl = V(a) - V(b)$

① We say $F : \mathbb{R}^d \to \mathbb{R}^d$ is a potential force if $\exists$ a C^1 fn $V : \mathbb{R}^d \to \mathbb{R}$ $\ni$ $F = -\nabla V$

② We say $F : \mathbb{R}^d \to \mathbb{R}^d$ is a conservative force if $\forall a, b \in \mathbb{R}^d$, $\forall$ two paths Γ_1 & Γ_2 joining a & b, $\int_{\Gamma_1} F \cdot dl = \int_{\Gamma_2} F \cdot dl$.

(i.e. the line integral of F along a path only depends on the end points of the path & not the path itself).

Physics:



Note: A potential force is necessarily cons. ind of path (only depends on end pts).

(Pf: Say $F = -\nabla V$. Then $\int_\Gamma F \cdot dl \overset{\text{FThm}}{=\!=} V(a) - V(b)$)

Claim (IOU): A conservative force is a potential force (if the domain is simply connected (has no holes)).

(Claim (IOU): Cons $\iff$ potential $\iff$ irrotational

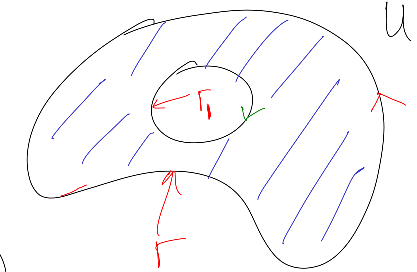
$$\underbrace{\overbrace{\nabla}^{}\times F}_{\substack{\text{curl}\\ \text{(IOU).}}} = 0 \quad \text{(IOU)}$$
)

Greens theorem:

Greens theorem: (1) $U \subset \mathbb{R}^2$ bounded (nice) region

∂U = Boundary of U

Assume ∂U = Finite union of piecewise C^1 closed curves.



(In picture $\partial U = \Gamma \cup \Gamma_1$

exterior boundary interior bdry)

Orientations: Γ = exterior bdry oriented counter clockwise

Γ_1 = int bdry " clockwise.

Let $F : \bar{U} \longrightarrow \mathbb{R}^2$ be C^1 $(\bar{U} = U \cup \partial U)$

Then $\displaystyle\underbrace{\int_{\text{bdry of } U \to \partial U} F \cdot d\ell}_{\text{line int}} = \underbrace{\int_U (\partial_1 F_2 - \partial_2 F_1)\, dA}_{\text{area int}}$

$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

Goal: Prove Greens thm

↳ 2 parts

(1) prove it on a square.

(2) Use coordinate changes to prove on arb domains.

Warning: (1) Thm fails if U is not bdd.

② Thm fails if F is not C^1 on all of U

$$\frac{1}{2\pi}\int_{\Gamma} \frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{1}{2\pi}\int \nabla\theta \cdot d\ell$$

$$\frac{1}{2\pi}\left(\theta(b) - \theta(a)\right)$$